

THE Mathematics Student

*A Quarterly Dedicated to the service of Students and Teachers
of Mathematics in India.*

VOL. VI.—1938

Edited by

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Annamalai University, Annamalainagar, South India

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and others.

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List of exchanges with the Publications of the Indian Mathematical Society

India

1. Proc. of the National Institute of Sciences of India.
2. Trans. of the National Institute of Sciences of India.
3. Proc. of the Indian Academy of Sciences.
4. Current Science.
5. Bulletin of the Calcutta Math. Society.

Japan

1. Collected Papers from the Faculty of Science, Osaka Imperial University.
2. Japanese Journal of Mathematics.
3. Journal of the Faculty of Science, Hokkaido Imperial University
4. Proc. of the Physico Mathematical Society of Japan.
- Tohoku Mathematical Journal.

England

1. Monthly Notices of Royal Astronomical Society.
2. Proc. of Cambridge Phil. Society.
3. Proc. of the Edinburgh Mathematical Society.

4. Journal of London Mathematical Society.
5. Mathematical Gazette.
6. Math. Notes of the Edinburgh Math. Society.
7. Science Progress.
8. Quarterly Journal of Mathematics.

Sweden

1. Acta Mathematica.
2. Arkiv fur Matematik Astronomi ok Physik.
3. Lund University Journal.

Holland

Nieuw Archief Voor Wetkunde.

Belgium

Sphinx.

France

1. Annales de l'Institut Henri Poincaré.
2. Annales Scientifiques de l'Ecole Normale Supérieure.
3. Bulletin des Sciences Mathématiques.
4. Comptes Rendus de l'Académie des Sciences.

A Quarterly Devoted to the Science of Statistics and Progress
of Mathematics in India.

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ANARASINGA KAO MAI

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1939

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THE MATHEMATICS STUDENT

Volume VI]

MARCH 1938

[Number 1

THE TENTH CONFERENCE OF THE INDIAN MATHEMATICAL SOCIETY

The Tenth Conference of the Indian Mathematical Society met in Lucknow on the 15th, 16th, and 17th of March 1938, under the auspices of the Lucknow University.

The Address of Welcome to the delegates of the Conference was read by Dr. R. P. Paranjpye, Vice-Chancellor, Lucknow University, on the evening of the 15th March. The Conference was then inaugurated by the Hon'ble Pandit Govind Vallabhai Pant, Premier of the United Provinces. The Secretary of the Society, Dr. Ram Behari, then read the Report of the Society for the years 1936-38. The Inaugural Address was then delivered by Dr. R. Vaidyanathaswami, Reader in Mathematics in the Madras University and Hony. Joint Secretary of the Society.

At 5-30 P.M. Dr. R. P. Paranjpye was at home to the delegates and to the members of the Reception Committee.

At 6-30 P.M. the Hon'ble Justice Sir Shah Sulaiman, Judge of the Federal Court, Delhi, delivered a popular lecture on "The Theory of Gravitation."

On Wednesday the 16th March, the session commenced at 1-30 P.M. and was devoted to the following Symposium Lectures :

Sir Shah Sulaiman : On his own modification of Einstein's Equations.

Dr. A. C. Banerji : Shapes of Nebulae

Dr. V. V. Narlikar : Gravitational Situations

Dr. T. Vijayaraghavan : Tauberian Theorems

Dr. F. W. Levi : Permutations of enumerable aggregates.

At 5-30 P.M. Prof. J. A. Strang, Head of the Mathematics Department, Lucknow University, was "at home" to the delegates and members of the Reception committee.

An hour later Dr. A. C. Banerji, Head of the Mathematics Department, Allahabad University, delivered a popular lecture on "Stars and Galaxies"

On Wednesday the 17th March, a discussion on the teaching of Mathematics in Schools and Colleges commenced at 10-30 A.M.

This was followed by a business meeting of the members of the Indian Mathematical Society to consider suggestions regarding the efficient working of the Society.

The Reading of papers was taken up in the afternoon.

At 5-30 P.M. the Reception Committee was "At Home" to the Delegates.

The popular Lecture for the evening was delivered by Dr. T. Vijayaraghavan.

The conference dissolved on the evening of Thursday 17th March after a hearty vote of thanks to the Chairman and members of the Reception Committee, to the hosts at the several "at homes", to those who had delivered the popular and Symposium lectures and to the volunteers who had helped in the arrangement of, what was acknowledged on all hands, as a very successful session of the Indian Mathematical Conference.

Reception Committee

Chairman

Dr. R. P. Paranjpye, M.A., B.Sc., D.Sc., Vice-Chancellor,
Lucknow University.

Vice-Chairman

Prof. J. A. Strang, M.A., B.Sc., Head of the Mathematics Department
Lucknow University.

Secretaries

Dr. A. N. Singh, M.Sc., D.Sc., Lucknow University.

Prof. N. N. Bose, M.Sc., Lucknow University.

Treasurer

Dr. Lakshmi Narayan, M.A., D.Sc., Lucknow University.

WELCOME ADDRESS

BY

DR. R. P. PARANJPYE, *Vice-Chancellor, Lucknow University.*

GENTLEMEN: I have very great pleasure in welcoming you to Lucknow for the first time for this session of the Indian Mathematical Conference, and I trust that your short stay here will be comfortable and useful. The city of Lucknow is perhaps the most beautiful in the whole country, and although perhaps it cannot boast of an antiquity like Delhi or Benares, it has played a prominent part in the history of India during the last two hundred years. Some of its monuments and buildings will amply repay a visit, even from mathematicians.

2. The University of Lucknow has only a short history and although it cannot claim any very large share in the advance in our science that is going

on throughout the country, I hope that this visit will encourage us to take our proper share in the development of Mathematics. As in other Universities this subject is taught here up to the M. A. and M. Sc. standards, and we have produced a certain number of students with very good academic record. A few have been engaged on mathematical research and have published a certain number of papers, and I have every hope that the distinction which the University has gained in several other branches of learning will also be soon attained in this branch of Mathematics.

3. Lucknow has been famous in India as the centre of Islamic culture, and although this culture is more in the direction of poetry and letters, still it cannot be forgotten that in ancient days Mathematics formed a prominent part in the history of Hindu and Muslim civilisations. The Arabs, as is well known, formed the connecting link between the Mathematics of the Greeks and the Mathematics of Ancient India, and several Sanskrit works on Mathematics became first known to European scholars through the medium of translations of Arabic renderings of Sanskrit works. It is not, I hope, too much to expect that the present-day culture of Lucknow will be amplified by the addition of mathematical and scientific studies.

4. It is not possible in this welcome address to speak about any abstruse branch of Mathematics, and your subsequent sittings and the lectures arranged will give you ample opportunities of discussing such matters. The Indian Mathematical Society has been doing its best to promote Mathematical studies for over twenty-five years by maintaining a Library and publishing its journals. This work ought, of course, to be continued and extended. But if an *Indian* Mathematical Society is to have any special feature which it is the only competent body to develop, I would respectfully urge that it is in the direction of a detailed study of ancient Indian and Arabic literature in Mathematics. During the last one hundred years many scholars have studied this part of the history of Mathematics, and books on the History of Mathematics like those of CANTOR and BALL devote certain chapters to this subject. But I have always felt that a systematic plan for this purpose is required, and if this Society can formulate that plan and work it strenuously, it will be a great feather in its cap and will, moreover, fill an important *niche* in the history of our science. A member of the Mathematical staff of this University has, in collaboration with another scholar, published the first volume of the History of Indian Mathematics, and I understand that the second volume is soon going to be published. I congratulate the authors on this work, and hope that they will carry it to completion in a short time. I would, however, very much like to see that most of the material for such a history is properly published so as to be available for future workers in the same subject. As is well known, most of the works of the Greek mathematicians are available to the students of the history of Greek mathematics. I should like the same thing in the case of Indian and Arabic mathematics. For this purpose various important works in these languages should be published containing their text, extracts from commentaries on them, their translations, and notes. For bringing out such editions,

the editors should be competent scholars of the languages concerned as well as students of Mathematics. Perhaps, it may not be possible to get many people with this double qualification, and each work might have to be the result of collaboration between two authors, one of whom would be a Sanskrit or Arabic scholar and the other a mathematician. But with the large number of Universities in this country it should not be difficult to get a sufficient number of interested people to undertake this work, which ought to be mainly a labour of love. The editors will have, probably, to do the work without much remuneration. But the expenditure of publishing these editions should be borne by bodies like the Universities, learned Societies, and even perhaps, by Governments in the country. Each volume will probably require not more than three or four thousand rupees, and if two or three volumes are brought out every year, in a few years India will have carried out a work of supreme importance to the History of Mathematics, and have repaid its duty to its ancient *savants*. I do hope that this Society will take the lead in the matter and put itself into touch with other bodies for drawing up a plan for this work. Perhaps, the project I have mentioned may not appear to be exactly within the range of the Indian Mathematical Society, but I have always held the view that the study of the history of a subject ought to be an integral part of the study of that subject itself.

5. I do not propose to delay you any longer. It is a very great pleasure to me to welcome you all here, and in particular to welcome the Hon'ble Pandit Govind Ballabh Pant, the Premier of the United Provinces, who has kindly consented to inaugurate this Conference. Mathematics, perhaps, does not come exactly within the purview of a busy politician and administrator, although even he cannot utterly neglect the subject which has a large bearing on such practical branches of administration as Economics, Statistics, Life Insurance, etc. But no one who has the interest of the people at heart can neglect such an essential avenue of culture as Mathematics. It has always occupied a prominent place in the educational system of every country in the world and will continue to do so in future, in spite of occasional remarks about the alleged waste of time in the study of such a barren subject as Mathematics. If the people of a country are to be properly educated, we claim for Mathematics an essential rôle in that education. I therefore have great pleasure in requesting the Hon'ble the Pandit to inaugurate the Conference.

Message from the Minister of Education, United Provinces

Ministry of Education, United Provinces,
Lucknow.

March 14, 1938.

DEAR DR. SINGH,

I should have very much liked to be present at the forthcoming conference of the Indian Mathematical Society, but I very much regret to say that I have to be out of station during the days the Conference will be in session. I have been a humble student of Mathematics having had the privilege of being a

student of the late Dr. Ganesh Prasad, and try to keep myself in touch, with what success it is not for me to say, with the latest advances in the Science. It is a matter of sincere gratification to me that India's contribution to this branch of Science, a contribution which the world has a right to expect from us, looking to our past, is increasing both in volume and quality as the years pass.

I wish the Conference every success.

Yours sincerely,

DR. A. N. SINGH, M.Sc., D.Sc.,
Secretary, Reception Committee,
Indian Mathematical Society,
Lucknow University.

SAMPURNANAND.

Secretary's Report for the Period December 1935 to March 1938

BY

DR. RAM BEHARI, M.A., Ph.D. *Hony. Secretary.*

MR. PRESIDENT, LADIES AND GENTLEMEN :

I. On behalf of the Managing Committee of the Indian Mathematical Society, I have the proud privilege to extend to you all, a hearty welcome to this Conference. We are profoundly grateful to the members of the Society and to other distinguished mathematicians, who have come from all parts of the country to attend this Tenth Conference of our Society, which is intended to promote the advancement of Mathematical study and research. We are equally grateful to the many citizens of this historic city of Lucknow, who have responded to our invitation by gracing this occasion with their presence.

We are specially grateful to you, Sir, for the trouble you have taken, in spite of your multifarious duties to come to inaugurate our deliberations.

Lastly I take this opportunity of giving expression to the grateful thanks of the Society to the various Local Governments, Native States, Universities and Institutions for the facilities they have granted to their officers to enable them to attend this Conference.

2. First of all it is my painful duty to record the heavy loss which the Society has sustained in the deaths of :

(i) V. Ramaswami Aiyar, M.A., who founded the Society in 1907 and rendered valuable services in various capacities ; who was a great source of inspiration to our younger members, and whose love and devotion to the subject is embodied in the large number of interesting papers, notes and questions, that have appeared in our Journals. Those who attended the last Conference held at Delhi must be remembering him well.

(ii) Dewan Bahadur R Ramachandra Rao, B.A., an ex-president of the Society for two years (1915-1917), who was instrumental in discovering and helping the late Srinivasa Ramanujan, F.R.S.

(iii) Rao Bahadur S. Narayana Iyer, M.A., who was an enthusiastic member of the Society from its very inception, Assistant Secretary (1907-1910); Honorary Treasurer (1914-1928); It was he who helped in securing Ramanujan a post in the Port Trust Office which gave him facilities to continue his research

(iv) Prof. V. B. Naik, M.A. of Fergusson College, Poona, Honorary Assistant Librarian (1907-1922); Honorary Librarian (1922-1937), through whose selfless work and paternal care the Library of the Society grew considerably.

(v) Prof. S. Mukhopadhyaya, M.A., Ph.D. of Calcutta University, who took an active interest in the affairs of the Society, and was a member of the Managing Committee at the time of his death.

3. Turning to routine matters, during the time that has elapsed since we last met at Delhi in December 1935, about 30 new members have been enlisted, including one Life Member. Prof. G. N. Watson, Prof. E. H. Neville and Dr. Vaidyanathaswamy have during this period kindly accepted our invitation to be enrolled as Honorary Members of the Society.

4. The Society continues to publish two journals, each of which appears four times a year. The first, which retains the appellation, the 'Journal of the Indian Mathematical Society,' publishes only original papers and caters for the needs of workers in the field of advanced Mathematics. It is under the fostering care of Dr. Vaidyanathaswami of the Madras University. The other one under the title 'The Mathematics Student' is edited by Prof. A. Narasinga Rao of the Annamalai University and meets the requirements of Students and Teachers of Mathematics in Colleges, and of young research workers.

5. The Library of the Society is flourishing under the care of Prof. R. P. Shintre and Prof. D. D. Kosambi, both of the Fergusson College, Poona. A most welcome feature is the increase in the number of journals received in exchange. In 1933, their number was only 29, while now it has gone up to 62. This change has been brought about by the efforts for our Librarians and the co-operation of various mathematicians abroad. In particular, we acknowledge our gratitude to Prof. Tullio Levi-Civita of the University of Rome, who gave a great deal of his precious time and used all his influence in having the *Rendiconti* of the Lincei Academy exchanged with our publications.

The number of journals subscribed by the Library has also increased appreciably. Some important publications like the *Mathematische Zeitschrift*, *Zentralblatt für Mathematik*, *Composito Mathematica*, have been recently added to the list.

Due to the increase in the number of journals, the space which the Library Hall of the Fergusson College could afford, proved insufficient for the needs of the Library. The books were, therefore, shifted to a room on the ground floor

of the same building, open steel shelves extending up to the ceiling were specially constructed, and space has been provided not only for the journals in stock, but also for those likely to come in for several years more.

6. Regarding the circulation of journals, I take this opportunity of soliciting the co-operation of the members who wish to take advantage of the circulation privilege in making this part of the work of the Library free from delays, mistakes and losses.

7. With regard to the financial position of the Society, every year several members are taking advantage of the reduced rate of life composition to become life members, with the result that the annual income from ordinary membership fees is gradually going down. The Managing Committee hopes that with the willing co-operation of all the members, the deficit will be made up by enlisting new members, by persuading members in arrears to pay up promptly, and by enlisting more institutional subscribers for the journals. So far the Government of India have not seen their way to grant us the financial assistance sought from them. The Committee hopes that in view of the Society's pressing needs for the housing and funds for the library, the Government of India and the Universities will come to our aid by making suitable annual grants.

8. This Conference should in the usual course have been held during the last Christmas, which perhaps, would have suited our members, especially those from the South better, but in view of the Session of the Indian Science Congress early in January, it was not considered suitable by our hosts. It will serve to show the interest evinced by our Members in these periodical Conferences if I mention that the number of papers has grown steadily from 13 read in the first Conference held at Madras, to 40 in the last Conference held at Delhi. This time, the number of papers contributed is even more. We have also been able to arrange Symposium lectures on different research topics.

9. In conclusion, the Committee, once again, takes this opportunity to express its feelings of thankfulness for the trouble taken by you, Sir, to come here in the midst of your onerous duties, to initiate the deliberations of this Conference. The Committee also has great pleasure in offering its thanks to all those who have been ungrudgingly contributing their mite for the welfare of our Society, during these 31 years of its existence, and especially to those who have undertaken long and tedious journeys to attend this Conference and to contribute towards its success.

The Managing Committee also thanks the various Indian Universities, that have rendered financial help in the past by giving annual grants-in-aid.

My last but not the least duty is to express, on behalf of the Managing Committee, our grateful thanks to the Vice-Chancellor of the University of Lucknow, for giving us this opportunity of assembling here under the auspices of the University by his kind invitation, and to the Local Reception Committee for their hospitality and excellent arrangements for the guests.

Our special thanks are due to Dr. A. N. Singh and his band of volunteers who have worked hard to make this Conference a success.

STARS AND GALAXIES

BY

PROF. A. C. BANERJI, D.Sc., *Allahabad.*

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Gentlemen, You have seen stars and nebulae or at least you are acquainted with them. You may also have observed the Milky-way on a dark clear night. It appears as a faintly luminous band forming an arch which stretches across the sky from end to end. If you see it through a good telescope you will perhaps think that it is composed of "countless multitudes" of stars. But the number is not really uncountable. The total number is of the order of two hundred thousand millions *i.e.*, 2×10^{11} . Each star is an immense fiery globe similar to our Sun. The number of human beings on our earth is about two thousand millions. So the number of starry inhabitants in the Milky way is about two hundred times the human population on our planet. All these stars form a system or a galaxy whose size is enormous but not unlimited. As we are situated within the system, we cannot get a good view of its form. An inhabitant of Andromeda Nebula, or more generally a person situated outside the Milky-way will see it as a little island of matter in an ocean of empty space, just as we find the Andromeda nebula to be a tiny island in the ocean of emptiness. Or we may put it in a slightly different way and say that both the Milky way and the Andromeda nebulae are oases in the desert of emptiness. Although they are mere oases in empty space, their sizes are enormous compared to us. To measure their dimensions it is most inconvenient to employ our ordinary units of distances, and we have to use instead our astronomical units. The usual units that are used are one light year and one parsec. One light year is the distance covered in one year by light travelling with the speed of 186,000 miles per sec. One light year is roughly equal to 6 billion miles (6,000,000,000 miles). One parsec is the distance of a heavenly body whose parallax is one second. *i.e.*, it is about 20 billion miles (20,000,000,000 miles).

According to most recent estimate by Hubble and Trumpler, the diameter of the Milky way is about 30,000 parsecs. Lindblad's theoretical treatment gives 26,000 parsecs as the diameter. Our galactic system or the Milky way is a greatly flattened disc with the thickness at the centre of about 6000 parsecs, Our sun is at a distance of 10,000 parsecs from the centre of mass of the galactic system. This centre is situated in the Constellation of Ophiuchus. By studying the motions of stars in our galaxy, it has been found that our galactic system is a dynamical unit revolving in its own plane round the centre like a huge Catherine wheel. The period of rotation is about 220,000,000 years, and the tangential velocity at the

sun is about 200 miles per sec. (275 Km/Sec.). The total mass of the galaxy is about 165×10^{10} times the mass of the sun. The largest known external system (external to our galaxy) is the Andromeda nebula. According to most recent estimate it has a diameter of about 20,000 parsecs. Shapley thinks that it is also a greatly flattened stratum of about 1000 parsecs in thickness except near the galactic centre where it is greatly enlarged. Its period of rotation is 14,000,000 years. Its mass is about 3.5×10^9 times the mass of the sun. Hubble has shown the practical identity of the constituent members of the Andromeda nebula and our galaxy. Each contains stars and star clouds, bright and dark nebulae, diffuse nebulae, globular clusters, cepheid variables novae, giant and super-giant stars. The similarity of structure of the two systems is strongly indicated by their greatly flattened forms. Both are in rapid rotation in their planes and this rotation is perhaps the cause of their flattening. The Andromeda nebula has the form of a spiral and it is not unreasonable to suppose that our galaxy is a great spiral nebula, perhaps somewhat larger than any of the extra galactic system, but not of a different order of size. The Andromeda nebula and the Milky way are super galaxies, and several super galaxies form together a meta galaxy. If we take the analogy of terrestrial geography then a *meta galaxy* may be compared to a *continent* and a *super galaxy* to a *country* within a continent. "Know thy country well" is the slogan at present and we shall therefore study more in detail the constituents of our galactic system *i. e.*, the Milky way.

I already mentioned to you that the Milky way contains "multitudes of stars". These stars individually show a very wide diversity. Some radiate more than 10,000 times as much heat and light as our sun while some others give out less than 1/100th times as much as our sun. Some show a surface temperature as $30,000^\circ \text{C}$. while others have got surface temperature as low as $3,000^\circ \text{C}$. These stars have widely varying degrees of apparent brightness due (i) partly to their distances and (ii) partly to their actual differences in intrinsic brightness.

Some stars are believed to be pulsating. They undergo inflation and deflation within a period of a few days or a few weeks. They give out widely varying quantities of heat and light as they swell up and collapse. A great many stars, about $\frac{1}{3}$ of the whole number, travel in pairs and they are called double or binary stars, but the majority of them to quote the language of Eddington, are "bachelors" like our sun. Some stars are very dense and compact, while others are extremely tenuous. The stars also vary enormously in size.

But in spite of their diversity in many respects there is some sort of uniformity amongst the stars so far as their masses are concerned. But for a few exceptional ones the masses of stars vary within a range from 1/5th to 5 times the sun's mass. The general run of masses is even within a narrower range. Adult persons have widely different features, but their masses will generally vary within the narrow range of 1 md. and $2\frac{1}{2}$ mds.

A giant star is big not on account of its mass but because it is swollen up like a balloon. Some of these stars are as rarefied as gas. A dwarf star is

small because it is highly compressed. Our sun is a typical middle class star. The mass of our sun is 1.99×10^{33} gm., its radius is 6.9×10^{10} cm. and its mean density is 1.4 times that of water. The surface temperature of the sun is about 5000° C. whereas the temperature at the centre may be as high as $14,000,000^{\circ}$ C.

Capella which is the brightest star in the constellation Auriga is a binary star. Its bright component has a mass 83×10^{33} gm. or $418 \times M$ where M is the mass of the sun. Its radius is 955×10^{11} cm. or $13.74 \times R$ where R is the radius of the sun. Its mean density is .00227 gm. per cu. cm. So Capella has an average density about the same as the air in our atmosphere. Its surface temperature is about 5200° C. while the temperature at its centre is estimated to be over 7 million degrees ($7,000,000^{\circ}$ C.).

Betelgeuse is a real giant star in the constellation of Orion. To get an idea of its size we have to imagine our sun to swell up and extend up to the orbit of Mars, swallowing up Mercury, Venus, and the Earth. Its radius is 2.5×10^{14} cms. or $360 \times R$. The mean density is .000002 gm. per cu. cm. Therefore the matter in Betelgeuse is extremely tenuous. It is as rarefied as the gas in an exhausted vessel for instance--a vacuum tube. For terrestrial conditions, Betelgeuse may be regarded as "a rather good vacuum", but when we consider the extreme emptiness of space Betelgeuse can no longer be regarded as a vacuum, it then becomes an aggregate of matter—a star. Recently it has been possible to produce a more highly exhausted vacuum than Betelgeuse in our laboratories. If a star of the size of Betelgeuse has as high a density as the sun, then the force of gravity will be so great that light would be unable to escape from it and rays would fall back to the star like a stone to the earth.

In gaseous nebulae we observe matter which is a million times more rarefied than the most highly exhausted vacuum obtainable on earth. In our laboratories we can hardly produce temperatures more than $5,000^{\circ}$ C., but in stars we some time come across temperatures as high as $50,000,000^{\circ}$ C. Eddington has very aptly stated that in the laboratories of the heavens we come across much hotter furnaces and much more rarefied "vacuum"; and a vaster scale of experiments is in progress there.

White dwarf Stars

Eddington has compared an astronomer to an electric engineer. The electric engineer will tell you that to produce a certain amount of illumination, you must have a dynamo of a specified size. Similarly it is possible for an astronomer to calculate the mass of a star which gives out a certain amount of light and heat. Eddington thus found the "mass-luminosity law" of a star *i. e.*, the intrinsic brightness of a star depends mainly on its mass and to a small extent upon its radius. This formula was not intended to apply to all stars but only to diffuse stars which behave like perfect gases. The observed masses and luminosities of diffuse stars agreed with the formula all right, but the trouble arose when dense stars for which the formula was not

intended, agreed with the formula equally well. This surprising result needed explanation. It meant that although the density was as great as water or iron the stellar material was behaving like a gas and was compressible like an ordinary gas. We have to look into this point more carefully. Let us see why we can compress air and not water to any appreciable extent. It is because that the air molecules are far apart with plenty of empty space between them. When air is compressed the molecules are packed a bit closer, then the amount of vacant space between them is reduced. In liquids and solids the molecules are practically in contact and the limit of compression has already been reached so that the particles cannot be packed closer together any further; and the densities of these liquids and solid substances can no longer be increased appreciably. So under terrestrial conditions the limit of compression corresponds approximately to the density of the liquid or the solid state.

But stellar conditions where temperatures of millions of degrees are reached, are very different from conditions prevailing on our earth. At such high temperatures atoms are thoroughly smashed up or ionised. Under ordinary terrestrial conditions an atom has a heavy central nucleus surrounded by widely extended but not very substantial structure of revolving electrons. At the high temperature prevailing in the stars this structure of electrons is broken up and they are scattered loose. Vast extent of empty space is now available; the atoms do not jam, and the limit of compression is not reached until a density very much greater than that of water or indeed of any material found on this earth, is attained. Stellar matter is far from the limit of compression which has been put enormously higher due to the smashing up of the outer parts of the atom. So this matter behaves like a perfect gas.

The companion of Sirius, the brightest star visible in the sky, has a density 50,000 times that of water—the matter in this star is 2000 times as dense as platinum. If a match box is filled with this matter, it would weigh a ton. The star α Eridani B has a density about 98,000 times that of water.

Planetary Systems

At one time it was thought that all stars have attendant planets. Now the opposite view is taken and it is supposed that planetary system is a rare development and a "freak-formation." The formation of planets of our sun is a rare accident. It is surmised that some two thousand million years ago our sun nearly collided with another star. There was no actual smash. The second star has passed by and is far away now. A great tidal wave was created in our sun at the time of the closest approach, and jets of matter were raised and thrown out, which have condensed into planets now. There is immense room in inter-stellar space, and Eddington has rightly pointed out that notwithstanding the reckless speed of stars, the celestial traffic conditions are extremely safe, and collisions are very rare occurrences. It has been calculated that with the present distribution of stars in the neighbourhood of the sun, a given star is likely to meet with actual collision once in 6×10^{17} years.

If we allot an average age of 5×10^{12} years to the stars, we find that only about one star in 100,000 is at present surrounded by planets.

Binary Stars

The two constituents of some of the binary stars are so closed together that they cannot be differentiated by direct visual observation. Only their spectra reveal that they are binary stars. They are called *spectroscopic binaries*. They are susceptible to direct visual observation. It is surmised that spectroscopic binaries are formed out of rotating masses of nearly uniform density.

The parent rotating mass which had originally a spherical shape assumes first the spheroidal and then the pear-shaped form through shrinkage. The pear-shaped form is dynamically unstable and ultimately fission occurs, and two detached masses are formed, which make a binary system.

On the other hand the visual binaries are formed out of independent condensations in the parent nebulae.

Cepheid Variables

Cepheid Variables are perhaps pulsating stars. They are very useful in measuring the distance of remote stars and nebulae. Parallax method of measuring distances fails when a heavenly body is at a distance greater than 100 parsecs. Brightness or luminosity of a Cepheid variable undergoes continuous change periodically, and the period varies from a few hours to a few weeks. Cepheids which have the same period, have the same intrinsic luminosity, radius and spectral type. The observed relation between period and luminosity is governed by *period luminosity law*. The intrinsic luminosity of the Cepheids varies more or less as the period. So Cepheid Variable can be made to act as "Standard candles." A Cepheid having a period of 40 hours has an intrinsic luminosity 250 times that of the sun and a Cepheid having a period of 10 days has got a luminosity 1600 times that of the sun. When the absolute and apparent magnitudes of a Cepheid are known a simple application of the inverse square law will give its distance. If A and B have the same intrinsic brightness, then if A looks four times as bright as B, B must be at a distance twice that of A.

Luckily for us, most of the star clusters and nebulae abound in Cepheid Variables, and so their approximate distances are easily found.

Planetary and diffuse nebulae

Planetary nebulae have nothing of a planetary nature in them, but they appear to have finite discs when seen through a telescope. These nebulae contain a great number of stars. They are very much rarefied. Eddington has calculated that the whole amount of matter contained in a volume of the size of the earth would not weigh more than 20 tons.

Diffuse nebulae are irregular in shape. Variation of density, opacity, and luminosity is responsible for all sorts of fantastic shapes. In our galactic system we come across both these types of nebulae. Some of them have a diameter of 100 light years or more. If our galactic system is compared to a *country* on the same analogy these nebulae can be compared to *provinces* on our terrestrial map.

There are also dark nebulae which do not shine and which obscure the stars which lie behind them. The Pleiades and the Hyades are galactic clusters which are much smaller in size. The length of the Pleiades is about 10 light years from end to end. These galactic clusters may be compared to *districts* and *divisions*. On the same scale the sun with its planets may be compared to a *city* with its *suburbs*.

There is another kind of cluster called the globular clusters which are much richer in stars than galactic clusters.

Magellanic clouds and extra-galactic nebulae

Just outside the boundary of our galaxy there are two remarkable groups of stars which were first observed by the famous Spanish Explorer Ferdinand Magellan near the South celestial pole when circumnavigating the world. Their distances are about 85,000 and 95,000 light years from us. In the depths of space there are many extra-galactic systems or nebulae, and out of these a large number have spiral forms while others have got elliptical shapes. The great nebula Andromeda is about 800,000 light years from us. Many of these extra-galactic nebulae have got enormous size. If the size of some of these nebulae be reduced to that of Asia, then the Earth would become a microscopic body hardly visible even under a most powerful microscope. Many of the spiral nebulae have spiral forms.

Extra-galactic nebulae abound in Cepheid Variables and their distances can therefore be easily measured. We notice one remarkable thing about these extra-galactic nebulae. Except for three or four of the nearest of them their spectra show a shift of spectral lines towards the red. It is moreover found that on the average the amount of the shift is directly proportional to the distance of the nebulae. Perhaps the most probable explanation is to be given by Doppler's Principle. In accordance with this principle we find that these galaxies are running away from us and the further they are away, the faster they go. The speed is found to be proportional to the distance, so that a galaxy 10 million light years away recedes at the rate of 1000 miles per second, a galaxy over 50 million light years away recedes at the rate of 5000 miles per second. The fastest yet discovered recedes at the rate of 30,000 miles per second. Its distance is 300 million light years away from us. In miles, 300 million light years = 1800,000,000,000,000,000 miles.

It may be mentioned here that in the case of a nebula we cannot get a good clear spectrum like that of a star. This is due to the fact that a nebula

consists of multitudes of stars with all sorts of different spectra which are superposed. We therefore get a sort of a blur in which two main features common to all the stars are shown prominently. These are H and K lines of Calcium, and the shifts of these lines are measured, from which we can deduce the velocity of recession of the nebulae.

The supporters of the theory of Relativity have tried to explain the phenomenon of recession of galaxies by suggesting that it is due to general scattering caused by *cosmical repulsion*. In all small scale systems gravitation gets the upper hand whereas in a large enough scale cosmical repulsion preponderates. In accordance with this theory the nebulae will recede to double their present distance in 1300 million years.

There is one point which I would like to mention to you. We can only know ancient, indeed, very ancient history of the nebulae. A nebula whose distance is observed to be 10 million light years from us was at that distance 10 million years ago. At the present moment it is somewhat further away i.e., about 10,050,000 light years away from us, and is receding from us at the rate of 1050 miles per sec. Similarly a nebula whose distance is observed to be 300 million light years from us was at that distance 300 million years ago. At the present moment it is about 350 million light years from us and is receding now at the rate of 35,000 miles per second.

I am indebted to Eddington, Jeans, Shapley, Hubble, Lindblad and others for borrowing data from their papers in preparing this lecture.

ABSTRACTS OF PAPERS

F. W. LEVI Hardinge, Professor of Mathematics,
Calcutta University

Permutations of enumerate sets

P. SAMUEL LALL, Punjab University
Dedekind's Sections of real numbers and Mahajani's variation of it.

S. M. SHAH, Muslim University, Aligarh
An inequality for an arithmetical function.

S. S. PILLAI, Annamalai University
'On Waring's Problem' $g(6) = 73$.

As usual, let $g(n)$ denote the least value of s such that every number is the sum of s n^{th} powers. Then for $n \leq 100$, we do not know the exact value of $g(n)$ only when $n=4, 5$ and 6 . In this paper I show that

$$g(6) = 73$$

The proof rests on certain results quoted by L. E. Dickson from a table prepared by R. Shook. While no table is required to determine $g(n)$ when $n \geq 7$, we want Shook's extensive table to determine $g(6)$. Though a proof which does not depend upon an extensive table is highly desirable, the author wonders whether it will be possible unless one finds out a method which is entirely different from all existing ones.

K. SAMBASIVA RAO, M.A. (Hons.), Andhra University.

On a particular representation of integers as sums of k^{th} powers.

We represent an integer x in the form $x = x_1^k + x_2^k + \dots + x_r^k$ where $x_1, x_2, \dots, x_r$ are integers given by

$$(x_1 + 1)^k > x \geq x_1^k; (x_2 + 1)^k > x - x_1^k \geq x_2^k \text{ etc.}$$

If the number of k^{th} powers required for representing x is $S_k(x)$ it is shown that

- (i) for all $x > x_0$, $S_k(x) < c_1 \log \log x$;
- (ii) there exist infinitely many x for which $S_k(x) > c_2 \log \log x$.

HANSRAJ GUPTA, Government College, Hoshiarpur.
Congruence Properties of Self-Contained Balanced Sets.

A set B_k of positive integers $b_1, b_2, b_3, \dots, b_k$ is called a balanced m -set, when the sum of the b 's equidistant from the beginning and the end is m . It is said to be self-contained modulo m , when numbers ξ exist such that the members of set ξB_k are congruent modulo m with the members of set B_k , each to each though not necessarily in the same order.

Denoting by $G_r(B_k)$ the sum of the products taken r at a time of the members of set B_k , and by $S_r(B_k)$ the sum of the r^{th} powers of b 's, the following results are proved :—

$$2(m, M_{2j}) G_{2j+1}(B_k) \equiv 0 \pmod{m^2},$$

$$\text{and } 2(m, M_{2j}) S_{2j+1}(B_k) \equiv 0 \pmod{m^2},$$

where M_{2j} is a certain product quite independent of m .

HANSRAJ GUPTA, Hoshiarpur
A Generalisation of Leudesdorf's Theorem.

Denoting by G_r the sum of the products taken r at a time of numbers less than and prime to m , the following generalisation of Leudesdorf's Theorem is proved :

$$2(m, 4N_{2j}) G_{2j+1} \equiv 0 \pmod{m^2}$$

where N_{2j} is the product of all odd primes of the form $(2j'd + 1)$.

HANSRAJ GUPTA, Hoshiarpur
Waring's Problem for powers of Primes.

In this paper, decompositions into 12th powers of primes are considered. The weight of any integer $\leq 11^{12}$ can be found with the help of tables given. Finally it is shown that the weight of no number between 7^{12} and 10^{12} exceeds 1670.

D. R. KAPREKAR, Devlali
Demlo Numbers and their applications to recurring decimals.

A Demlo number is an integer whose digits may be divided into three groups of which the middle group consists of a repeated digit and the sum of the number given by the two extreme groups is also a number with the same repeated digit. Thus 12488764 is a demlo number since $124 + 764 = 888$. Methods of deriving such numbers, their properties and relation to recurring decimals are discussed. A special class of such numbers are those in which the digits go on increasing and then decrease e.g. 123454321. They have some striking properties ; thus

$$12345678987654321 \times 243 = 29999999994000000003.$$

A. A. KRISHNASWAMI AYYANGAR, M.A., L.T.
Maharaja's College, Mysore.

A New Study of the half-regular continued fraction.

In this paper it is proposed to consider the structure and properties of the several types of half-regular continued fractions through a study of the transformations which convert a simple continued fraction into a half-regular one. This method enables us to give a synoptic survey of known results and many new ones concerning the half-regular continued fraction. The discussion runs into five sections :

- (1) Sets of Transformations.
- (2) Properties of Residues.
- (3) Varieties of Types.
- (4) Raschesten Kettenbruch-entwicklung.
- (5) Recurring continued fractions,

the main aim being to bring together as many results as possible under one unifying principle.

T. VENKATARAYUDU, M.A., Andhra University, Waltair.

The Algebra of the e^{th} power residues.

In the present paper the algebra of the Gaussian periods is shown to be identical with the algebra of the e^{th} power residues mod. p . The latter is easily seen to be a particular case of the algebra of the e^{th} power residues connected with a cyclic group of finite order. Further an example is given to show that there is no corresponding algebra of the e^{th} power residues connected with any finite Abelian group.

S. PANKAJAM, University of Madras.

On finite Boolean Algebra

This paper is concerned with a proof of the known theorem that any finite Boolean Algebra can be regarded as an algebra of all sub-classes of a finite class. The proof, though not new in principle, has, I believe, some novel features. Further, I also discuss the representation of a finite Boolean Algebra as a Vector Algebra, (mod. 2).

C. N. SRINIVASIENGAR

Non-differentiable functions

M. N. NARASIMHA IYENGAR, Mysore.

A General form of the Taylor-Remainder :

It is proposed to show that by utilising Cauchy's form of the mean value theorem it is possible to obtain a general form of the

Remainder in Taylor's theorem, which includes as particular cases all the forms of the Remainder known so far, including the Integral forms. A short note on the behaviour of ' θ ' is also included.

M. N. NARASIMHA IYENGAR, Mysore

'Some Inequalities connected with the Error-Function Integral and the Gamma Integral.'

The approximate position of the Centre of Gravity of the area is used to derive these inequalities. One of them is

$\frac{4}{a} \frac{1 - e^{-a^2/2}}{1 + e^{-a^2/2}} > \int_0^a e^{-x^2/2} dx > \frac{2}{a} (1 - e^{-a^2/2})$. This inequality can be made stronger: As it is, the portion $\int_0^a e^{-x^2/2} dx > \frac{2}{a} (1 - e^{-a^2/2})$ seems to be better than $\int_0^a e^{-x^2/2} dx > \sqrt{\pi}/2 (1 - e^{-x^2}) (1 > x > 0)$ in the interval $(0, .95)$.

M. N. NARASIMHA IYENGAR, Mysore

'Some simple properties of the Logarithmic curve and their applications.'

The locus of the mid-points of a system of parallel chords is obtained and it is shown how all the variables that occur in a paper by Prof. G. N. Watson on 'the approximations connected with e^x ' (published in the Proc. L. M. S.) can be interpreted geometrically.

M. N. NARASIMHA IYENGAR, Mysore

'A theorem stated by Ramanujan'. (Approximations connected with e^x).

The following theorem was stated by Ramanujan:

$1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} y = \frac{e^x}{2}$, where $y = \frac{1}{2}$ when $x = 0$ and $y \rightarrow 1/3$ when $x \rightarrow \infty$ and he remarked that it is difficult to prove that y lies between $\frac{1}{2}$ and $\frac{1}{3}$. Professors Watson and Szego have published proofs in *Proc. L. M. S.* and *Jour. L. M. S.* respectively. In this paper an attempt is made to give an elementary proof of the same result.

V. GANAPATHY IYER, Madras

"On the average radial increase of integral functions of order one and finite type."

Let $z = re^{i\theta}$. Let $f(z)$ be an integral function and $M(r, f) = \max_{|z|=r} |f(z)|$. We denote by $n(r)$ the number of zeros of $f(z)$ in $|z| \leq r$.

and by $n(r, \alpha, \beta)$ those in the region $\alpha \leq \theta \leq \beta$, $|z| \leq r$. The function $f(z)$ is said to be of order one and finite type when $\lim_{r \rightarrow \infty} \frac{\log M(r, f)}{r} < \infty$. In this paper I prove the following result :

THEOREM. Let $f(z)$ be of order one and finite type and let

$$\int_1^\infty \{ \log^+ |f(x)| + \log^+ |f(-x)| \} \frac{dr}{r}$$

be convergent, where $\log^+ |a| = \max(0, \log |a|)$. Then the following relations are valid :

- (i) $\lim_{r \rightarrow \infty} \frac{n(r, \delta, \pi - \delta)}{r} = \lim_{r \rightarrow \infty} \frac{n(r, -\pi + \delta, -\delta)}{r} = 0$ for $0 < \delta < \pi$;
- (ii) there exists a constant A such that $\lim_{r \rightarrow \infty} \frac{n(r)}{r} = A$;
- (iii) $\lim_{r \rightarrow \infty} \frac{n(r, -\delta, \delta)}{r} = \lim_{r \rightarrow \infty} \frac{n(r, \pi - \delta, \pi + \delta)}{r} = \frac{1}{2} A$ for $0 < \delta < \pi$;
- (iv) $\lim_{R \rightarrow \infty} \frac{1}{\log R} \int_1^R \left\{ \log |f(t e^{i\alpha})| + \log |f(t e^{i(\alpha - \pi)})| \right\} \frac{dt}{t^2}$
 $= \pi A |\sin \alpha|$ for $0 \leq \alpha \leq 2\pi$.

Results (i), (ii) and (iii) have been proved by Levinson* and M. L. Cartwright.† The relation (ii) has also been proved by Paley and Wiener.‡ The result (iv) is new.

R. S. VARMA, Cawnpore

Some Infinite Integrals involving Weber's parabolic cylinder functions.

The object of this paper is to evaluate some infinite integrals involving Weber's parabolic cylinder functions. Two functions of the parabolic cylinder are also obtained which are J_n -transforms of each other. Finally some integral expressions for the Humbert function $J_m, n(x)$ are investigated.

R. S. VARMA, Cawnpore

On Laguerre Polynomials which are self-reciprocal in the Hankel transform

In this note some Laguerre polynomials which are self-reciprocal in the Hankel transform have been investigated.

* *Proc. Cambridge Phil. Soc.*, 31 (1935), 335-46.

† *Ibid.*, 347-50 and the *Proc. Lond. Math. Soc.*, (2), 38 (1935), 179.

‡ "Fourier Transforms in the Complex Domain", *Amer. Math. Soc. Colla. Pub.*, 19 (1934), 69.

B. RAMAMURTI, Annamalai University

A special web of binary sextics

The object of this paper is to study the special web of binary sextics

$$A \beta_x^2 \gamma_x^2 \delta_x^2 + B \alpha_x^2 \gamma_x^2 \delta_x^2 + C \alpha_x^2 \beta_x^2 \delta_x^2 + D \alpha_x^2 \beta_x^2 \gamma_x^2$$

where $\alpha_x^2 \beta_x^2 \gamma_x^2 \delta_x^2$ are general quadratics, by means of a representation in [3] with reference to a twisted cubic. Surprisingly enough it leads to the net of quadrics associated with a pair of Möbius tetrads, which has been studied in detail recently by W. L. Edge (*Proc. London Math. Soc.*, 1936). Some important properties of such a net obtained by him, and some others are deduced in this paper by binary methods.

A. NARASINGA RAO, Annamalai University

Oriented Turbine Geometry

The oriented turbine is a generalisation of the concept of the oriented sphere and is a configuration of oriented surface elements obtained as follows:—Let the surface elements of an oriented sphere (inner sphere or hub of the turbine) in n -dimensions glide along the tangent prime in all directions through a constant distance t from the point of contact. The points of the resulting configuration of oriented surface elements lie on a sphere (the outer sphere or rim of the turbine) and make with its surface elements a constant angle α . When the rim and the hub of a turbine coincide we have an oriented sphere which is thus a special kind of turbine. It is shown how the Lie Group in Höhere-Kugelgeometrie may be considered as a Geometry of Turbines under a contact transformation group, and how not only the Geometries of Möbius and of Laguerre but a whole pencil of sub-geometries of Lie which belong to the Möbius-Laguerre pencil may be imbedded in Turbine geometry.

A. NARASINGA RAO AND K. RANGASWAMI,

Annamalai University

On the Pedal and Contact Circles of a Triangle

A pair of isogonal conjugates P, P' with respect to a triangle define two circles—their “pedal” and “contact” circles. The former passes through the projections of these points on the sides, while the latter is the circle through the points of contact of the two in-conics with centres at P and P' . In this paper these circles are studied through the usual representation of the circles in a plane by the points in a three-space.

It is known that the totality of pedal circles and also that of contact circles correspond to points on quadrinodal cubic surfaces, the in- and ex-circles of the triangle corresponding to the nodes. It is shown that the collineation which carries the contact circle of a pair isogonal conjugates into the pedal circle of the same pair is the one which has the in- and ex-circles of the triangle for fixed elements and carries all straight lines in the plane into circles orthogonal to the polar circle of the triangle.

A. NARASINGA RAO AND B. R. VENKATARAMAN,

Annamalai University

The infinitesimal flow in a projective space

When an infinitesimal projective transformation is conceived as a flow of points on the plane supposed endowed with a Euclidean metric, it is found that the divergence and curl of the flow vector at a point P are proportional to its distances from two fixed lines at right angles passing respectively through the centroid and ortho-centre of the invariant triangle. Taking the two lines as the x and y axes, it is found that the velocity field may be resolved into (i) a constant translatory motion, (ii) that due to a homogeneous strain of zero divergence and (iii) a radial outflow along OP with a velocity ry where O is the origin and $OP=r$.

The projective flow in Euclidean n -space shows the same features, the Stokes tensor here taking the place of the curl of the vector field.

C. N. SRINIVASIENGAR

On the Quartic Developable

Deals with a family of Cayley cubic Scrolls associated with a quartic developable, all of whose generators belong to the same linear congruence.

The first polar surface of the developable with respect to any point on a generator is a cubic of the type (B_5, C_1) , whose asymptotic curves are determined.

RAM BEHARI, University of Delhi

Some properties of the line of the striction of a ruled surface

The object of this paper is to obtain both geometrical and analytical proofs of some properties of the line of striction of a ruled surface, and to give an easy method for constructing a ruled surface whose line of striction is any given curve.

RAM BEHARI, University of Delhi

On the generators of a ruled surface

Mainardi-Codazzi relations are generally employed by writers to obtain some properties of the generators of a ruled surface. In this paper I have obtained an easy method for proving some properties without any reference to Mainardi-Codazzi relations.

D. D. KOSAMBI, Fergusson College, Poona

Differential invariants connected with systems of partial Differential Equations: By using the methods followed in my work on ordinary differential equations, it is possible to obtain the differential invariants of the system: $\frac{\partial^q x^i}{\partial u^{\alpha_1} \dots \partial u^{\alpha_q}} \cdot H^i_{\alpha_1 \dots \alpha_q} = 0$. The results

obtained differ somewhat from those of Bortolotti and also of Kawaguchi. One fact is clear: the *natural* connection is not affine but projective. Moreover, it cannot be determined intrinsically without recourse to the second partial derivatives of the $H^i_{\alpha_1 \dots \alpha_q}$.

R. VAIDYANATHASWAMI, University of Madras

On the space of line-elements in the calculus of variations

K. NAGABHUSHANAM, Andhra University, Waltair

On curves minimising a certain integral.

If g_{ik} be metrical ground form in Euclidean 3-space, the degenerate tensor

$$G_{ik} = x^r x^s (g_{rs} g_{ik} - g_{ri} g_{sk})$$

is connected with the varying part of moment of inertia in the theorem of parallel axes. If we inquire into the nature of the curves minimising $\int ds$ where $ds^2 = G_{ik} dx^i dx^k$ we obtain as solution

$$\frac{x^1}{l} = \frac{x^2}{m} = \frac{x^3}{n}, \quad (l, m, n \text{ being arbitrary constants}).$$

From physical considerations, it is shown that these straight lines through the origin which are null extremals are the only curves in the solution.

K. NAGABHUSHANAM, Andhra University, Waltair

On a class of degenerate metrics

The nature of the metrics which are necessarily degenerate, whose only extremals are null extremals forming a single family of curves is studied here,

K. NAGABHUSHANAM, Andhra University, Waltair

Variational principle in a nonkinematical case of Milne's Expanding Universe.

Milne's equations for trajectories in Expanding Universe of 4 dimensions are not derivable from a regular metric as shown by Kosambi in the Kinematical case where $G \equiv -1$. Following the same notation and methods, the tensors P^i_j , R^i_{jk} for general $G(\xi)$ are calculated and found to be as follows:

$$X^2 P^i_j = GX(Y\delta^i_j - \dot{x}^i \dot{x}_j) + Q_j \left(p^i \Gamma - \dot{x}^i G' \frac{Z}{Y} \right),$$

$$X^2 R^i_{jk} = X(G - G'\xi)(\delta^i_j \dot{x}_k - \delta^i_k \dot{x}_j) + p^i \Gamma(p_j \dot{x}_k - p_k \dot{x}_j) + G'Z(\delta^i_j p_k - \delta^i_k p_j),$$

where

$$Q_j = Y p_j - Z \dot{x}_j \quad ; \quad G' = \frac{dG}{d\xi} \quad ; \quad G'' = \frac{d^2 G}{d\xi^2} \quad ;$$

$$\Gamma = -G(2+G) + G'(1+G)(2\xi-1) + (G')^2 \xi(1-\xi) - 2G''(1+G)\xi(1-\xi).$$

In the case when $G \equiv$ a constant $\neq -1$, it is found that if a metric exists, then

$$YA[-(1+G) + \xi(2+G)]$$

is also a metric, A being independent of $\dot{x}^i$. As it is not possible to find such A, it is concluded that in this case also no regular metric exists.

M. V. VAIDYANATHA SASTRI, Nizamiah Observatory

Occultations of stars observed at the Nizamiah Observatory

The occultations observed at the Nizamiah Observatory from 1927-1937 are dealt with in this paper. The method of reduction is essentially that outlined by Innes in A. J. 835 and for convenience of reference is briefly explained. The reduction of occultations yields equations of the form

$$\sigma' - \sigma = \Delta\lambda \cos(\rho - \chi) + \Delta\beta \sin(\rho - \chi)$$

where

σ' = apparent angular distance of the star from the centre of the tabular moon.

σ = apparent semi-diameter of the moon.

ρ = Direction of the moon's motion measured from the north point.

χ = Position angle of occultation measured from the north point of the moon's disc in the direction N.E.S.W.

$\Delta\lambda$ and $\Delta\beta$ are the errors in the orbital longitude and latitude of the moon.

The solution, by the method of least squares, of equations of the above type for each year gives the correction to be applied to the mean longitude and latitude of the moon.

The values for $\Delta\lambda$ as reduced from the above observations are found to be in close agreement with Brown's as reduced from a very large material.

B. R. SETH, University of Delhi

Waves in a circular Canal

No exact solution exists for the problem of the transverse oscillations of water contained in a canal of circular section. When the free surface is at the level of the axis both Lamb and Rayleigh have obtained an approximate value for the frequency of the gravest mode by using Rayleigh's method for cases in which the normal types cannot be accurately determined; but both of them have failed to notice that the constrained type assumed by them is unstable, and hence the result obtained is of not much consequence. In the present paper it is shewn that for a circular canal no exact stable type can be obtained. But, if the depth of water in the canal be small compared with its diameter, an approximate determination of the frequency can be effected. For very small depths the formula

$$\sigma = \left(\frac{g}{a}\right)^{\frac{1}{2}} \left(1 + \frac{5h}{2a}\right)$$

holds good, h being the depth of water in the canal.

B. R. SETH, University of Delhi

Transverse waves in canals

In this paper it is shewn that no exact solution can be obtained for transverse oscillation in a canal whose section is a closed curve. In certain cases when the depth of water in the canal is small an approximate value of the frequency can be obtained. This value approaches $\sqrt{g/\rho}$ as the depth becomes very small, ρ being the radius of curvature at the lowest point.

M. ZIAUD DIN, M.A., Ph.D. (Wales), Bahawalpur

Wave mechanics of the Harmonic Oscillator and the parabolic cylinder functions

The Methods of Quantum mechanics, considering the well-known Schrodinger's Wave equation, for the Harmonic Oscillator, are used to deduce the recurrence formulae for the Parabolic Cylinder Functions.

M. RAZIUDDIN SIDDIQI, Osmania University, Hyderabad

"Matricial representation of quaternions, and its application to relativistic quantum mechanics"

In continuation of the work of Kwal and others, an attempt has been made to represent the Hamiltonian quaternions as matrices. This is done by means of three sets of four matrices each, viz. e_μ, f_μ, g_μ ($\mu=1, 2, 3, 4$), corresponding to the three products $q \bar{p}$, $\bar{p} q$ and $q p$, where $\bar{p}$ is the adjoint quaternion to p . All the e, f, g , are matrices of the fourth rank, e_4 and f_4 being simply the unit matrices. A few of the relations satisfied by them are

$$e_r e_s = -e_s e_r = e_t; f_s f_t = -f_t f_s = f_r; e_r^2 = f_r^2 = -e_4 = -f_4;$$

$$g_r f_s = e_r g_s = g_t; g_s f_r = e_s g_r = -g_t,$$

where r, s, t denote cyclic permutations of 1, 2, 3.

$$e_\mu f_\nu = f_\nu e_\mu = g_\mu g_\nu; f_\mu g_\nu = g_\nu f_\mu; g_\mu f_\nu = e_\mu g_\nu.$$

Quaternionic functions and derivatives can easily be expressed by means of these matrices. The D'Alembertian differential operator can be linearised, and Maxwell's equations expressed in a matricial quaternionic form. This matricial representation of the group of quaternions is then seen to be identical with the group of Lorentz transformations. These transformations then lead us directly to Dirac's equation in quaternionic form.

All the variables in quantum mechanics thus become quaternionic functions of the quaternionic parameter, viz. the world point. In this relativistic quantum mechanics we get four equations of motion corresponding to each equation of Heisenberg, and four uncertainty relations corresponding to each classical relation. The generalisation is analogous to that introduced by Dirac for Schrödinger's wave-equation.

B. S. MADHAVA RAO, Engineering College, Bangalore

Biquaternions in Born's Electrodynamics.

A representation of Born's original theory by the use of biquaternions is given by Watson (*Trans. Roy. Soc. Canada*, 1936). In this paper will be found alternative representations and also generalisation to the several types of Born's Electrodynamics.

M. R. DORESAMIENGAR, University of Mysore

'A Variation of Pareto's Law'

Pareto's law of incomes has been given as $N_x^* = A/x^a$. It can also be expressed as $\frac{dN}{N} / \frac{dx}{x} = -a$. This paper points out that a has to be

necessarily greater than 2 if the statement is to be significant at all. For $a < 2$ implies frequency-distributions of incomes with infinite standard deviations. But the *simplest* of variants of Pareto's law, namely, $\frac{dN}{N} / \frac{dx}{x} = -Kx$ points to distributions with finite S.D's and invariable Lorenz Coefficients. The paper also points out the economic inferences that can be drawn from the modified Pareto's Law.

S. SUBRAMANIAN, Annamalai University

Compatibility of Fisher's tests for Index Number Formulae

Out of the seven important tests devised by Prof. Irving Fisher for Index Number Formulae, Prof. Frisch found* three incompatible with one another; but as I have pointed out elsewhere† his conclusion is rendered incorrect by his failure to take into account the limitations of the general solution he gives for his differential equations.

In the present paper the tests are examined in detail and it is shown that a formula can be found so as to satisfy them simultaneously.

* Frisch: *Jour. Amer. Statist. Assoc.* (1930): p. 404.

† Subramanian: *do.* *do.* : p. 316.

GLEANINGS

The straight line offers a curious object for analysis. It is not for the eye a very easy form to grasp. We bend it or we leave it. Unless it passes through the centre of vision, it is obviously a tangent to the points which have analogous relations to that centre. The local signs or tensions of the points in such a tangent vary in an unseizable progression; there is violence in keeping to it, and the effect is forced. This makes the dry and stiff quality of any long straight line, which the skilful Greeks avoided by the curves of their columns and entablatures and the less economical barbarians by a profusion of interruptions and ornaments.

GEORGE SANTAYANA: *The Sense of Beauty.*

A fire mist and a planet,
 A crystal and a shell,
 A jellyfish and a saurian,
 And caves where the cave-men dwell
 Then a sense of law and beauty,
 And a face turned from the clod;
 Some call it EVOLUTION,
 And some call it GOD.

WILLIAM HERBERT CARRUTH.

Summary of Sir Shah Sulaiman's Popular Lecture

Newton's Mechanics was designed for a Superman, who is himself at rest and can observe distant bodies instantaneously and measure absolute space and time exactly. A poor human observer is not aware of his own exact motion through space and can only employ a messenger like light who travels with a finite velocity only. Nevertheless, Newton's Law of Gravitation had signal triumphs when after explaining the motion of the moon round the earth and of the earth and the other planets round the Sun it led to the discovery of unknown planets. The position of the planet Neptune in the Heavens was calculated by two astronomers independently from its effect on Uranus, and was observed at the predicted spot. As recently as 1930, Pluto was discovered as a result of calculations made by Lowell, though it is really doubtful whether on account of its small mass it could really have been predicted on those calculations.

But in some other respects Newton's Law has undoubtedly broken down, and has been dethroned by Einstein's Theory. But Relativity cannot be regarded as the last word on the subject. Relativity in its turn has certainly failed in at least two respects. First, according to it, there is in its Universe, so to speak, some cosmic form of repulsion which increases with the distance, that is to say, bodies repel each other more if they were further apart, an unnatural result. The consequence is that everything must be scattering away and heading towards swift disintegration. The whole universe is expanding or rather exploding away at a terrific pace. It was Dr. Hubble who definitely observed this effect and laid down what is called the linear proportionality law. But Dr. Hubble himself in his Rhodes Lecture at Oxford delivered in 1936 showed that an examination of the statistical observation of the nebulae demonstrates that there is no such expansion of the Universe at all. The apparent recession of the nebulae can be explained far more satisfactorily by an inherent loss of energy as light travels through the vast space during millions of years before reaching the earth.

Secondly, Rhoyd's photographs taken on the occasion of the total solar eclipse in June 1936, show that Relativity is 100% wrong in regard to the spectral shift from the edge of the Sun.

In a few months we may know whether Relativity is at all confirmed as regards the deflection of light from stars.

Sir Sulaiman's symposium lecture was illustrated by over a dozen slides showing how his theory lay midway between Newton's law and Einstein's Relativity, and how his equation was derived without Relativity postulates.

Discussion on the Teaching of Mathematics in Schools and Colleges

Thursday, the 17th March, 1938. 10-30 A.M.

Chairman:—DR. R. P. PARANJPYE

DR. T. VIJAYARAGHAVAN (Dacca) who initiated the discussion stressed the need for continual attention to the essentials of all mathematical reasoning, namely a steady logical progress from a set of hypotheses to a conclusion.

PROF. J. A. STRANG (Lucknow) while agreeing with the previous speaker pointed out that the average student is very far from having reached any such general view-point. For many mathematics remained an affair of water-tight compartments having little in common. One of the aims of every teacher should be to break down this isolation of different branches and weld Algebra, Geometry, Calculus and the rest into a powerful weapon of thought, characterised by a single method. As regards University instruction, the historical approach was valuable as a means of arousing and maintaining the interest of the student and widening his outlook.

DR. R. SIDDIQI (Hyderabad, Deccan) suggested the appointment of a commission to enquire into the Teaching of Mathematics in Indian Schools and Colleges, and suggest improvements.

According to DR. RAM BEHARI (Delhi) the aim of teaching should be utilitarian and cultural up-to the School Leaving stage, and here, there should be no insistence on rigour. At the Degree and Post-Graduate stages, the aim should be to give a sound training in the subject maintaining a modern standard of rigour, and to give ideas about mathematical discipline. He pointed out the need for basing the study of the subject on historical background so that the students might know something of the main facts of the mathematical past, and the great "heroes" of mathematics. He also suggested that some mathematicians should use mathematical methods in the elucidation of problems in Social sciences and other fields of study, so that the subject might become more interesting and useful.

The part played by examples in the teaching of mathematics was discussed by DR. B. N. PRASAD (Allahabad) who thought that too much time was devoted to them. The primary purpose of examples was to elucidate the basic theorems and for this purpose the simplest examples were the best.

DR. A. C. BANERJI (Allahabad) considered three possible aims in teaching mathematics in the M. A. and M. Sc. classes. Preparation for competitive examinations should not be our aim. On the other hand efforts should be made to move the Public Service Commission to bring the syllabus of their examination into line with the University syllabuses. He felt that a merely cultural training could not be the *sole* aim of a mathematical education. The M. A. and M. Sc. courses should be modelled on modern and progressive lines

so as to enable the student to start research immediately after the course. As regards scholarships and fellowships, more of them should be made available and their duration should be extended beyond two years.

DR. B. R. SETH (Delhi) was of opinion that research in Mathematics demanded certain inherent qualities and talents and every one should not be encouraged to take to it. Many of those who were supposed to carry on research did so because of the stipends attached to the work.

Discussing the aims of teaching Mathematics, PROF. A. NARASINGA RAO, (Annamalainagar), said that though in the case of most of those who underwent a course in the subject the actual content of the instruction in the shape of propositions and proofs were later forgotten, there still remained an immaterial residuum in the shape of an acuter appreciation of logical relations and a maturer aesthetic sense in matters intellectual. This was indeed the most valuable result of a mathematical training and it was this which we should seek to develop.

DR. F. W. LEVI (Calcutta) was of opinion that in this country teaching was hampered by examinations. Scholarships were often given for too short a period to people who felt compelled by their scholarships to publish papers of little value. It was desirable to arrange for an interchange of scholars and lecturers of different Indian Universities.

DR. ZIAUDDIN (Bahawalpur) said that too much stress was being laid at present on pure theory. He was for greater emphasis on practical work, the use of the slide rule and facility in numerical computation including the use of small calculating machines.

PROF. D. D. KOSAMBI (Poona) read extracts from Prof. Hardy's Address to the British Association "The case against the Mathematical Tripos".

Winding up the debate, the Chairman DR. PARANJPYE said that in teaching any subject we should follow the historical method. Historically, results like Taylor's series and the Fourier's Expansion Theorem were enunciated and used before the exact conditions of their validity had been investigated. It would be going too far to say that every result should be taught along with the full minutiae of its conditions of validity and with rigorous proofs. In conclusion he remarked that the object of teaching mathematics was not to produce mathematicians but good reasoners from given data.

Names of Delegates who were present at the Conference

Abhayankar, S. A., Gwalior.	Ram Behari, Delhi.
Banerji, A. C., Allahabad.	Rengaswami, K., Annamalainagar.
Banerji, Ekanath, Cawnpore.	Sambasiva Rao, K., Waltair.
Bhattacharya, D. P., Bareilly.	Seth, B. R., Delhi.
Dholakia, K. D., Bhavnagar.	Seth, S. D., Cawnpore.
Gupta, Hansraj, Hoshiarpur.	Shah, S. M., Aligarh.
Kaprekar, D. R., Devlali.	Shaikh, A. L., Karachi.
Kewalramani, M. K., Karachi.	Siddiqi, M. R., Hyderabad (Deccan).
Khanna, H. L., Cawnpore.	Sivasankaranarayana Pillai, S., Annamalainagar.
Kosambi, D. D., Poona.	Srinivasiengar, C. N., Bangalore.
Kureishy, A. M., Aligarh.	Srivastava, P. L., Allahabad.
Lall, Samuel, Lahore.	Subramaniam, S., Annamalainagar.
Levi, F. W., Calcutta.	Sulaiman, Sir S., Delhi.
Manerker, S. D.	Vaidyanathaswami, R., Madras.
Mehrotra, B. M., Benares.	Varma, R. S., Cawnpore.
Narasinga Rao, A., Annamalainagar.	Venkatarayudu, T., Waltair.
Narlikar, V. V., Benares.	Vijayaraghavan, T., Dacca
Pendse, G. G., Baroda.	
Prasad, B. N., Allahabad.	

THE INDIAN MATHEMATICAL SOCIETY

Statement of Accounts for 1936

Receipts.		Rs.	A.	P.	Expenditure.		Rs.	A.	P.
<i>Opening Balance :</i>					<i>Administrative Charges :</i>				
Savings Bank	...	...	1415	10 1	Secretary	...	...	91	0 0
Current account	...	...	195	9 6	Librarian	...	...	511	0 6
Post Office Cash Certificates	...	...	5600	0 0	Treasurer	...	...	20	0 0
Fixed Deposit	...	...	500	0 0	Assistant Secretary	...	...	269	1 9
<i>Subscriptions :</i>					Journal printing	...	...	1141	3 3
Life Members	...	...	350	2 0	Periodicals	...	...	406	11 0
Ordinary Members...	...	...	554	10 0	Bank Charges	...	...	5	4 10
Concession Members	...	...	735	12 0					
Subscribers	...	...	728	9 0					
<i>Journal :</i>					<i>Closing Balance :</i>				
Sale of journals	...	...	281	15 0	Savings Bank	...	...	2453	2 10
Sale of reprints	...	...	4	0 0	Current Account	...	...	304	13 2
<i>Grants-in-aid :</i>					P. O. Cash Certificates	...	...	5600	0 0
Madras University (35-36)	...	...	150	0 0	Fixed Deposit	...	...	500	0 0
Do. (36-37)	...	...	150	0 0					
Oosmania University	...	...	200	0 0					
Bombay University	...	...	200	0 0					
Annamalai University	...	...	100	0 0					
Interest on Savings Bank account	...	...	61	1 9					
Interest on Fixed Deposit	...	...	75	0 0					
			11,302	5 4				11,302	5 4

Checked and found correct.

28—2—1938.

(Sd.) G. A. SRINIVASAN.

28—2—1938.

(Sd.) L. N. SUBRAMANIAM,
Honorary Treasurer.

THE INDIAN MATHEMATICAL SOCIETY

Statement of Accounts for 1937

	Receipts.	Rs. A. P.	Expenditure.	Rs. A. P.
<i>Opening Balance:</i>				
P. O. Cash Certificates	...	5600 0 0	<i>Administrative Charges:</i>	
Savings Bank	...	2453 2 10	Secretary ...	49 8 3
Current Account	...	304 13 2	Treasurer ...	10 0 0
Fixed Deposit	...	500 0 0	Librarian ...	210 12 6
			Assistant Secretary	183 15 3
<i>Subscriptions:</i>			Joint Editor	25 0 0
Life Members	...	250 0 0	<i>Journal printing</i>	1109 4 0
Ordinary Members...	...	415 4 0	<i>Periodicals</i>	536 7 0
Concession Members	...	455 4 0	<i>Bank Charges</i>	4 6 0
Subscribers	...	627 6 3		
<i>Journal:</i>			<i>Closing Balance:</i>	
Sale of journals	...	242 8 0	Savings Bank	2352 5 9
			Do.	517 2 2
<i>Grants-in-aid:</i>			Current Account	342 2 2
Bombay University	...	200 0 0	P. O. Cash Certificates	5600 0 0
Ooemia University	...	131 2 3	Fixed Deposit	500 0 0
Delhi University	...	61 7 1		
Interest	...			
		11,440 15 7		11,440 15 7

Checked and found correct.
28--2--1938.

(sd.) G. A. SRINIVASAN.

28--2--1938.

(sd.) L. N. SUBRAMANIAM,
Honorary Treasurer.

INAUGURAL ADDRESS

BY

DR. R. VAIDYANATHASWAMI, *Madras University*

The Indian Mathematical Society started in 1907 for the advancement of higher mathematical study and research, by the late V. Ramaswami Aiyar, M. T. Naraniengar and a group of enthusiasts, has now had a career of some thirty years. During this period, it has performed as well as it could with the facilities at its disposal, its task of stimulating mathematical advancement, by publishing one journal which has lately become bifurcated into two, and by holding mathematical conferences once in two years or more seldom according to exigencies. In the first stages of the Society, its principal preoccupation lay in what is technically termed 'elementary geometry'; while much interesting and remarkable material can be read in the earlier numbers of the Journal, we do not find there a contact with the general spirit and the methods of modern mathematical thought. The present generation which has succeeded the generation of the founders, comprises several people with western training, and now the two journals run by the Indian Mathematical Society, as well as the journals conducted by its sister localised mathematical societies, may be said to reflect the growing assimilation of modern mathematics in this country. This is no doubt satisfactory and shews that we are not exactly stationary. But at the same time it has to be confessed that the ideal towards which we are striving is far from achievement; indeed it is not yet even in sight. This ideal is nothing less than the creative assimilation in this country of the whole of the modern scientific culture in its mathematical aspect, so that the fertile Indian mind with its untold potentialities may become seized of the new scientific situation, and resolve it in its own characteristic manner, in ways which it would be impossible to predict now. Mathematics is in a fundamental sense a master key to the scientific outlook of the present day, and the distance which separates us from our ideal may be estimated from the fact that the advanced pure and applied mathematical theories which are nowhere studied in Indian Universities well outnumber those which do receive any sort of study. At our present rate of progress, it will take a considerable period of time—certainly much more than a generation—before a

respectable level of mathematical culture can be reached in our Universities. The pace has therefore to be quickened; and the responsibility therefor devolves on the Universities, whose function on the one hand, is the active promotion of higher study and research, and on the other, the raising of the level of college instruction, so as to keep pace with the advance of research.

The explanation why Indian Universities are not as effective as they might conceivably be, in their double task of prosecution of research and its infusion into college instruction, can no doubt be sought in the troubled phase of political, social and cultural transformations through which we are passing, and the consequent stress of public activity and its diversion in various channels. But we are concerned here not with explanations, but with envisaging what has yet to be achieved, and figuring out the ways and means of doing it.

To achieve the full quickening of scientific knowledge based on the mathematical outlook, a necessary condition is the recognition of the high value of mathematical research, and of the young men in whom the research activity manifests. It is sometimes said that genius is born, not made, in such a way as to suggest that mathematical researchers are freaks of nature, endowed with an innate power of fulfilling themselves independently of social or environmental conditions. In so far as such statements may be made with the intention of disavowing responsibility for the proper utilisation and fulfilment of gifted young men, they are definitely false and misleading. Mathematics like all sciences is a social and institutional activity. Mathematical insight and sensitiveness to mathematical form, which are necessary for research, though not widely manifested, should not yet be called *rare*; we can indeed find several instances of them in every batch of students passing out from any of our Universities. The racial value of these qualities has to be appreciated and realised, and when one of these students takes up research, and proves himself capable of a new and original approach to mathematical ideas, he should be regarded as an asset to the race in the great struggle, which we vaguely indicate by the word 'civilisation'. Instead of damping his enthusiasm and putting obstacles in his path, institutional facilities should be so arranged as to draw him on from work to work, maturing and fully developing his faculties, and finally settling him in that life-work and career for which he is best fitted. That there are instances in India, where young men who have proved their worth in mathematical research have been forced to take to non-academic occupations, indicates that

a high sense of academic and human values in general, and of the value of mathematical research in particular, does not always inform the administration of the academic bodies in this country.

The Americans have developed to a high degree of perfection, their organisations of institutions for higher study, and their methods for stimulating research. In our present stage of scientific culture, we might do well in following up some of their methods. The art of administration consists in the judging of the available human material with reference to the ideal to be achieved, and the inducing it to put forth the best that it is capable of. I, for one, do not think that the best that the existing Indian mathematicians are capable of, is being got out of them. In the world of ideas, *publication* is the social symbol of final achievement; and though there are several Indian Mathematicians who have done substantial work in particular directions, we have not yet begun to write treatises or books on advanced mathematical subjects. If an academic body invites each mathematician to write a work on a prescribed subject, connected with his previous work, the carrying out of this task will not only fully develop the writer himself, but the publication of such works will be the first step in the building up of an indigenous mathematical tradition. Our own society could of course follow in the footsteps of the American Mathematical Society, and undertake this work and issue Colloquium publications, if it only possessed the necessary funds; pending the acquisition of funds, this task must be left to the Universities. The cost would not be much, and could if necessary be made up by cutting down some of the expenditure on casual lectures which leave no permanent impress behind them; on the other hand, the advantages in the direction of popularisation of higher mathematics and incitement to higher study would be considerable.

Lastly, there is a group of subjects, usually classed as applications of mathematics, in which mathematical concepts and methods are guided and controlled by the nature and requirements of the particular subject-matter concerned. Historically, Mathematics has both influenced and been influenced profoundly by such applications. Subjects of this sort—like Mathematical Economics, Statistics and the theory of Probability, and the various mathematical theories evolved with reference to situations in Modern Physics—are junctions where diverse streams of knowledge meet and mingle, and are specially sacred to scientific discipline—and for the same reason that Prayāg is sacred. By immersion in such junction-subjects, the

intellect becomes purified and attains to clear vision. Universities should evince a partiality for such subjects on account of their double character, and the vivid realisations of the meaning of mathematics which they supply.

* One of such junction-subjects is 'the foundations of mathematics'. The fact that it is one of the subjects which have not been taught or studied in Indian Universities, is the justification of the meagre account of it which I shall attempt here. This subject is concerned with the systematic and critical account of the ideas and processes of reasoning involved in the derivation of the concept of real number. The study of these ideas and processes of reasoning will naturally have intimate relations with Logic and Metaphysics. Our ancient Indian philosophers were generally inclined to class 'number' in the category of 'gunas' or qualities, but did not seem to have been aware of the immense possibilities which lurked in the number-concept. The different views which one can take in regard to the foundations of mathematics raise fundamental philosophic issues, and can be profitably studied not only by mathematicians, but by philosophers and students of philosophy. As a general principle, philosophy should not be taught on abstract lines from books, but in relation with the living problems of the present; and the greatest philosophical problems of the current century definitely centre round the disputes about the foundations of mathematics.

The three main directions along which the theory of the foundations of mathematics has developed are Logistic or Logicism, Formalism and Intuitionism. The first or logistic view attempts to exhibit mathematics as a branch of logic. The attitude of mind of the logistic philosopher may be described roughly thus: 'Logic is the science of reasoning i.e. it is an account of the principles involved in arriving at true conclusions, starting from true premises. Mathematics most certainly consists of true propositions about the real world, which have been arrived at by rigorous deduction from true premisses. At first sight, however, it appears to contain elements which cannot belong to logic e. g. the concept of infinity in its various forms and applications. In fact, Poincaré asserted definitely that the intuition of the unending series of natural numbers, or in other words, *The principle of mathematical induction*, is the one extralogical element in mathematics, and that any attempt at proof of this principle involves a vicious circle. But it is open to us to enlarge and revise our logic, by accepting such fundamental notions as basic principles on a par with the law of contradiction.

When we do this, mathematics could be exhibited as a branch of logic, thereby reaching the status of a science whose conclusions are true and relate to the real world.'

Thus the revision and the enlargement of the scope of traditional logic were undertaken as a preliminary to the reduction of mathematics to logic. The first step was to symbolise logic, that is, to reduce it to symbols operated by definite rules, so that all the basic principles and elements required in the process of reasoning were enumerated and taken account of. The next step was to get beyond the limitations of Aristotelian logic. The two main defects of the Aristotelian scheme are:

- (1) It makes no distinction between two different kinds of relations between propositions, namely those which refer to the constituents—subject and predicate—of the proposition, and those which involve no such reference.

The theory of the latter type of relations may be described as the calculus of unanalysed propositions, or the molecular theory of propositions, and is usually treated first in books on mathematical logic under the title of Propositional or sentential calculus. The relations which are studied in this calculus are those expressed by the word *not*, and by the conjunctions *and*, *or*, *if*. If p and q are propositions, the proposition, "if p then q " is written $p \supset q$ (p implies q), and the implication $p \supset q$ is usually interpreted in the *weak* sense "*not- p or q* ." From this weak interpretation of *if*, it follows at once that a false proposition implies every proposition and a true proposition is implied by every proposition. Symbolising *and* and *or* by $\cdot$ and $+$, the axioms of the Propositional calculus as given in *Principia Mathematica* are:

$$\begin{aligned} X + X &\supset X \\ X \supset X + Y \\ X + Y &\supset Y + X \\ (X \supset Y) \cdot \supset \cdot \{ (Z + X) \supset (Z + Y) \} \\ \{ X + (Y + Z) \} &\supset \{ Y + (X + Z) \}. \end{aligned}$$

The theory of the former type of relations is met in Symbolic logic by the conceptions of *Variable* and *Propositional Function*, and the classification of propositions into *particular* and *universal* is achieved by introducing two quantification-symbols, namely the All-symbol and the existential symbol.

- (2) The second and more serious defect of the Aristotelian logic is that it considers in effect only subject-predicate types of propositions.

It is unable to give proper account of a reasoning in which relations are involved; e.g. of the reasoning in 'If there is a son, there must be a father.' In the *Principia mathematica*, propositional functions of a single variable, answering to the scope of classical logic, are treated first, and relations, which amount in substance to propositional functions of several variables, are treated later. But in modern symbolic logic, the word predicate is often used, both in affirming an attribute of a single subject, and in asserting a relation between several subjects.

It was the first task of logistic theory to resolve the notorious contradictions in mathematical reasoning, like the paradoxes of Russell, Burali-Forti and Richard. This was done by Russell's doctrine of types, according to which the possible values of the argument of a propositional function were restricted to a certain type. But this doctrine of types does not give a good account of the situation in mathematics. In mathematics, a bounded set of real numbers has an upper bound which is also a real number; now, each real number is a class of rational numbers, and the upper bound is therefore a class of classes of rational numbers, and this though it belongs to a higher type than the real number can yet be identified with a real number. To explain this 'completeness of the real-number system', the axiom of reducibility was invented, according to which there is a predicate of the lowest grade having the same extension as a predicate of any grade. This axiom seems to run counter to the very purpose for which the doctrine of types was invented, and its logical status is dubious.

On the basis of these ideas and principles, the logistic treatise '*Principia Mathematica*' claims to reduce the number and function-concepts of mathematics to pure logic. Some of the defects in the Principia scheme have been pointed out by later writers and alternatives have been proposed.

According to the formalist view, the central fact about mathematics is that mathematical symbols are capable of divergent interpretations, and must therefore be meaningless in themselves. Historically this was first realised in the case of geometry. Thus in Poincaré's picture of plane hyperbolic geometry, in which there are an infinity of non-intersecting straight lines, the word 'point' is interpreted to mean point within a circle C , and 'straight line' to mean arc of circle orthogonal to C , contained within C . With this interpretation all the axioms of the geometry are true, and naturally all the results are also true. Therefore the words 'point', 'straight line', 'inci-

dence', 'distance', 'angle' occurring in hyperbolic or any other geometry, must be regarded as *not* denoting any definite types of object in themselves. They receive of course a partial implicit specification through the relations expressed by the axioms, but otherwise the words have no content whatever. Hence also mathematical propositions have no content, and therefore, though they can be deduced from the axioms, cannot be spoken of as true in themselves.

If we develop this attitude of mind, the view at which we arrive is that mathematics is a mode of operation with symbols which have no content, i.e. which do not refer to anything in particular. Certain combinations of symbols are given beforehand called axioms; also certain rules for deriving new propositions from given propositions. All propositions which can be derived in this manner from the axioms are called *demonstrable propositions*. The whole of the formal structure may be compared to a system of signs or a language. What the significant combinations of symbols are to be, and what symbols are to represent a proposition must be decided by reference to the meaning of the symbols in a particular interpretation. These are analogous to the rules of syntax of the language. If the name 'mathematics' is restricted to the purely formal deduction of demonstrable propositions from axioms, these considerations depending on the meaning of the symbols in any interpretation must be given another name 'metamathematics' or 'Theory of Proof', or 'Theory of Deductive Systems'. In a large sense then, mathematics including both aspects may be described as the study of 'the formal structure of symbolic deductive systems'. This includes logic as well, as the propositional calculus, alone or with the calculus of functions, forms such a deductive system.

The most important structural qualities of a symbolic deductive system reposing on a set of axioms are determined by the *consistency*, *independence*, and *completeness* of the latter. A set of axioms is *consistent* if a proposition and its contradictory cannot both be deduced from them. If the set is inconsistent, then since $p \cdot \neg p \supset q$, every proposition is demonstrable; conversely, the existence of a single refutable proposition is a proof of the consistency of the system. The set of axioms is *independent* if none of the axioms is a consequence of the remaining; lastly, the set is *complete*, if every proposition (in the language supplied by the axioms) is either demonstrable or refutable. Metamathematics is largely concerned with the investigation of these properties of axiomatic systems. The calculus of propositions is an example of an axiomatic system, and its consistency can be directly

established by the construction of refutable propositions, by a method known as 'the method of truth-values'.

More generally the Entscheidungs problem, namely the problem of deciding in a finite number of steps whether a given proposition is demonstrable or refutable or neither from a given set of axioms, is the major problem of metamathematical theory. It has been solved only for the simpler types of deductive systems; e.g. the propositional calculus, and the Aristotelian type of function-calculus i.e. calculus of propositional functions of a single variable. No general method is known by which the problem can be solved for more complicated deductive systems, in particular for Arithmetic.

The parallel postulate occupies a special position with reference to the remaining axioms of Euclidean Geometry, in that it can be neither demonstrated nor refuted from them. The remarkable result has been proved by Gödel that the Principia Mathematica deductive system, and a large class of like deductive systems are similarly *incomplete*; i.e. that there are formulae of the system which are neither demonstrable nor refutable in the system. The proof depends on the same principle as Richard's antinomy. The domain of the variables in the Principia deductive system may without loss of generality be considered to be the natural numbers. On account of the nature of the deductive process, the aggregate of demonstrable formulae must be enumerable; it will not be difficult to construct an actual scheme for enumerating them, as is done for instance in Carnap's recently published book 'The Logical Syntax of Language'. These demonstrable formulae define propositional functions of a single variable, the aggregate of which must be likewise enumerable. Let such propositional functions be enumerated and let $R(n)$ denote the n^{th} such function. Since the domain of the variables is the totality of integers, the propositional function $R(n)$ defines a class of integers which may likewise be denoted by $R(n)$. Let K denote the class of integers n which do not belong to $R(n)$, i.e. of integers n such that $n \in R(n)$ is not demonstrable in the system. We can easily shew that the propositional function corresponding to K is constructible in the system itself; hence K is an $R(q)$. Then the proposition $q \in R(q)$ is neither demonstrable nor refutable in the system. For if it were demonstrable, it would be true and therefore q would belong to K . Therefore by definition of K , $q \in R(q)$ is not demonstrable in the system. If on the other hand it were refutable, then the negation of $q \in R(q)$ would be demonstrable; hence q does not belong to K and therefore belongs to $R(q)$. Hence both $q \in R(q)$ and its negation would be provable, which is impossible.

This theorem seems to shew that the usual formalist methods of proving consistency have their limits, and that incompleteness is the characteristic of most deductive systems. The formalist theory has done great service to mathematics through its investigation of general deductive systems and its perfecting axiomatic technique.

The differences between logistic and formalism are not serious enough to stand in the way of their being welded together; in actual practice the outlook of the mathematician partakes of the viewpoints of both of them. Intuitionism however stands apart from them in its insistence on certain truths of a deeper character, and its expression of them in a revolutionary and paradoxical form. To stress the dynamic and creative character of mathematics, as against the static view to which the logistic and formalistic theories point, Intuitionism asserts that the objects of mathematics do not exist in a pre-determined transcendent state of being independent of the human mind, but are actually created and shaped by the Intuition which perceives them. Hence against formalists Intuitionism would urge that mathematical science deals with concrete objects, namely the cardinal integers and the relations which arise from them, which are presented to and have been constructed by the Intuition. Against logistic which believes in the pre-eminence of logic, Intuitionism would urge that Mathematics requires no other discipline as its presupposition, not even Philosophy or Logic, but is on the other hand the presupposition for every other discipline including Philosophy and Logic; for the mathematical intuition is the primordial intuition which perceives the natural numbers, and which can hardly be distinguished from Time-consciousness itself. Intuitionism derives certain crucial doctrines from this general position, which may be described as *realistic* in contrast with the *idealistic* position of logistic and formalism. Thus it follows from the doctrine regarding the nature of mathematical objects that mathematical existence *means* or is equivalent to constructibility in intuition; it also follows that mathematical problems which have remained unsolved, resisting all attempts at solution, need not necessarily possess a solution. If this is accepted, we can no longer say, that every mathematical proposition must in itself be either true or false, apart from any question of its being *experienced* as true or false in our constructive intuition; in other words, we must give up the principle of the excluded middle. This result should be compared with the formalist conclusion of the existence of *incomplete* systems (Gödel's theorem); the existence of dubious propositions which is arrived at by the formalist, by the analysis of deductive systems, is

reached by the Intuitionist by a direct *tour de force*, involving the surrender of the principle of the excluded middle.

The denial of the principle of excluded middle in the case of infinite aggregates, leads to revolutionary alterations in the results of classical mathematical analysis, for example in the notions of real number, limit, and convergence of infinite sequences. Several of the fundamental theorems of classical theory of functions of a real variable have to be given up, as they cannot be proved without appealing to the principle of excluded middle. The new form of function-theory which can be arrived at is very much more complicated than the classical form.

- References (1) Black—*Nature of Mathematics* (International Library of Philosophy and Scientific Method).
- (2) Heyting—*Intuitionism und Beweistheorie* (Ergibnisse der Mathematik; Julius Springer.
- (3) Hilbert and Äckermann—*Grundzüge der Theoretischen Logik*—Julius Springer.
- (4) Russell and Whitehead—*Principia Mathematica* Vol. I.
- (5) Russell—*Introduction to the Philosophy of Mathematics*.

REVIEWS

MADHUSUDAN SARKAR and BENODE BEHARI DUTTA *Astronomy*
Revised Edition (Kamala Book Depot, Calcutta)

Price Rs. 5/-

The book under review is the outcome of an attempt by the authors to produce a suitable text-book on astronomy that will meet the requirement of college students in India preparing for University examinations. Astronomy in most Indian Universities is studied only as one of the subjects for a course in Mathematics, without much emphasis being given to the observational side of the science; and a text-book intended for such a course has necessarily to be written with this object in view. The present volume, although appearing to be only an addition to the number of text-books already available, possesses certain features which are likely to make it useful to students beginning the study of the subject. Considerable care has been exercised in the choice of material and the plan of presentation is uniformly good.

A commendable feature of the book is the inclusion of matter that usually forms part of descriptive astronomy such as the system of stars and nebulae, the spectroscope and its applications. The treatment, in this part of the book is rather too brief; and perhaps one may be permitted to remark that the descriptions are not clear in some places and the explanations have not been given with sufficient precision. A few mistakes are also met with; for example, it may be observed that in the classification of variable stars p. 265, the name β Lyrae variables seems to refer to Cepheids which form a very important class of short period variables. While explaining the phenomena of variation p. 266, no mention is made of the pulsation theory of Cepheids. The term "variable star clusters" in page 271 is apparently intended to mean the well-known "cluster variables" which are also Cepheids, a class of stars of great importance in connection with the determination of parallaxes of globular clusters and other remote objects. The word 'opticians' in the same page appears to have been used instead of 'astronomers'. There are some obvious misprints which need not be given here; it is hoped that these will be rectified in the next edition. If one may venture a suggestion, the authors may take the opportunity to revise the descriptive part of the book thoroughly and, if possible add reproductions of a few typical photographs illustrating the objects described.

The collection of questions and numerical problems given by way of exercise is sure to be helpful to the beginner in understanding the principles of the subject. The book can be recommended to college students taking a course of astronomy for degree examinations.

T. P. B.

GLEANINGS

The revolution of the corpuscle

A corpuscle once did oscillate so quickly to and fro,
 He always raised disturbances wherever he did go.
 He struggled hard for freedom against a powerful foe—
 An atom—who would not let him go.
 The aether trembled at his agitations
 In a manner so familiar that I only need to say,
 In accordance with Clerk Maxwell's six equations
 It tickled people's optics far away.
 You can feel the way it's done,
 You may trace them as they run—

dy by dy less dz by dz is equal $K \cdot dX/dt$

.

While the curl of (X, Y, Z) is the minus d/dt of the vector (a, b, c) .

Some professional agitators only holler till they're hoarse,
 But this plucky little corpuscle pursued another course,
 And finally resorted to electromotive force

Resorted to electromotive force.

The medium quaked in dread anticipation,

It feared that its equations might be somewhat too abstruse,

And not admit of finite integration

In case the little corpuscle got loose,

For there was a lot of gas

Through which he had to pass,

And in case he was too rash,

There was sure to be a smash,

Resulting in a flash

Then dy by dy less $d\beta$ by dz would equal $K \cdot dX/dt$.

.
.

While the curl of (X, Y, Z) would be minus d/dt of the vector (a, b, c)

The corpuscle radiated until he had conceived

A plan by which his freedom might be easily achieved,

I'll not go into details, for I might not be believed

Indeed, I'm sure I should not be believed.

However, there was one decisive action.

The atom and the corpuscle each made a single charge,

But the atom could not hold him in subjection

Though something like a thousand times as large.

The corpuscle won the day

And in freedom went away

And became a cathode ray.

But his life was rather gay,

And he went at such a rate,

That he ran against a plate;

When the aether saw his fate

Its pulse did palpitate.

And dy by dy less $d\beta$ by dz was equal $K \cdot dX/dt$.

.
.

While the curl of (X, Y, Z) was the minus d/dt of the vector (a, b, c)

A. A. R. From G. F. HULL'S
An Elementary Survey of Modern Physics.

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A NEW STUDY OF THE HALF-REGULAR CONTINUED FRACTION*

BY

A. A. KRISHNASWAMI AYYANGAR, *Mysore.*

In this paper we propose to study the structure and properties of the several types of half-regular continued fractions by means of the transformations which convert a simple continued fraction into a half regular one⁽¹⁾ of the type

$$b_0 + \frac{\varepsilon_1}{b_1 +} \frac{\varepsilon_2}{b_2 +} \dots \dots \dots \frac{\varepsilon_n}{b_n +} \dots \dots$$

where $|\varepsilon_n| = 1$, $b_n + \varepsilon_{n+1} > 1$, and b_n a positive integer > 1 , in final.

For convenience, we adopt the following notations :

$$(i) (b_0; b_1, b_2, \dots) \equiv b_0 + \frac{1}{b_1 +} \frac{1}{b_2 +} \dots$$

$$(ii) (b_n, \overline{b_{n+1}}, \overline{b_{n+2}}, \dots, \overline{b_{n+r}}, b_{n+r+1}) \\ \equiv \frac{1}{b_n -} \frac{1}{b_{n+1} -} \frac{1}{b_{n+2} -} \dots - \frac{1}{b_{n+r} +} \frac{1}{b_{n+r+1}}$$

$$(iii) (b_{[r]}) \equiv \frac{1}{b +} \frac{1}{b +} \dots \frac{1}{b +} (r \text{ terms})$$

(iv) $S_{r,n} \equiv b_r + b_{r+1} + \dots + b_{r+n-1}$, where $b_r, \dots, b_{r+n-1}$ are the successive partial quotients of the half-regular continued fraction

$$b_0 + \frac{\varepsilon_1}{b_1 +} \frac{\varepsilon_2}{b_2 +} \dots$$

§ 1. Sets of Transformations

1.1 E_1 -transformation.

By repeated application of

$$(\alpha) \frac{1}{y} = 1 - \frac{1}{1 +} \frac{1}{(y-1)} \quad (\text{traceable to Lagrange}^{(2)}) \quad \text{we can go on}$$

* Contributed to the Tenth Conference of the Indian Mathematical Society, 1938.

† This and other subsequent numbers in brackets appearing between consecutive lines relate to references at the end of the paper.

diminishing a partial quotient greater than 1 and increasing at the same time the number of partial quotients.

$$\begin{aligned}
 &\text{Thus, } (b_{r-1}, b_r, \dots) \\
 &= (b_{r-1} + 1, \bar{1}, b_r - 1, \dots) \\
 &= (b_{r-1} + 1, \bar{2}, \bar{1}, b_r - 2, \dots) \\
 &= (b_{r-1} + 1, \bar{2}, 2, \bar{1}, b_r - 3, \dots) \\
 &\dots \\
 &= (b_{r-1} + 1, \bar{2}_{[k]}, \bar{1}, b_r - k - 1, \dots), \quad (k < b_r - 1) \quad \dots \quad (1)
 \end{aligned}$$

This transformation can evidently be continued up to $k = b_r - 2$, or $b_r - 3$ according as b_r is a medial partial quotient greater than 1 or a final quotient greater than 2, k being zero † or a positive integer. We call it the E_1 -transformation. Here, the partial quotient b_r (> 1) is replaced by $k + 2$ new partial quotients, while b_r itself is reduced by $k + 1$, the succeeding partial quotients, if any, being unaffected.

The maximum number of additional partial quotients that can thus be introduced is $b_r - 1$ or $b_r - 2$ according as b_r is medial or final; there are also just as many ways of applying the E_1 -transformation to the partial fraction $\frac{1}{\tau b_r}$.

We further notice that

$$S_{r-1}, n - n \quad (= b_{r-1} + b_r - 2) \quad \dots \quad (2)$$

is invariant under this transformation.

This property, being true for every E_1 -transformation, is also true for any aggregate of E_1 -transformations.

1.11 E_2 , E_3 , and P-transformations.

When the E_1 -transformation is pushed to its extreme limit, we have

$$(i) \quad (b_{r-1}, b_r) = (b_{r-1} + 1, \bar{2}_{[b_r-3]}, \bar{1}, 2),$$

and (ii) $(b_{r-1}, b_r, b_{r+1}, \dots) = (b_{r-1} + 1, \bar{2}_{[b_r-2]}, \bar{1}, 1, b_{r+1}, \dots)$ called the extreme E_1 -transformations.

† When $k=0$, the terms $\bar{2}_{[k]}$ simply drop out.

We may now apply a variant of (α), viz.,

$$(\beta) \quad \frac{1}{1 + \frac{1}{y}} = 1 - \frac{1}{y + 1} \text{ to the last term in (i) and the terms } \frac{1}{+1} - \frac{1}{+b_{r+1}} \text{ in (ii), and obtain}$$

$$(b_{r-1}, b_r) = (b_{r-1} + 1, \bar{2}_{[b_{r-1}]}) \quad \dots (3)$$

called the E_2 -transformation when $b_r (> 2)$ is a final quotient and

$$(b_{r-1}, b_r, b_{r+1}, \dots) = (b_{r-1} + 1, \bar{2}_{[b_{r-1}]}, b_{r+1} + 1, \dots) \quad \dots (4)$$

called the E_3 -transformation when $b_r (> 2)$ is a medial partial quotient.

The transformations (3) and (4) we call P-transformations when $b_r = 2$ (5)

It may be noted that the difference between the number of the partial quotients and their sum is increased by 1 on account of the E_2 - or the final P-transformation and by 2 on account of the E_3 - or the medial P-transformation.

1.12 New Convergents

From (3), we have

$$(b_{r-1} + 1, \bar{2}_{[k]}) = (b_{r-1}, k + 1), \text{ so that}$$

$(b_0; b_1, b_2, \dots, b_{r-1} + 1, \bar{2}_{[k]})$ is the $(k + 1)^{th}$ intermediate convergent between the principal convergents of the $(r - 2)^{th}$ and r^{th} orders.

Again, $(b_{r-1}, 1) = (b_{r-1} + 1)$, and

$$(b_{r-1} + 1, \bar{2}_{[k]}, \bar{1}) = (b_{r-1} + 1, \bar{2}_{[k-1]}, \bar{1})$$

$$\dots \dots \dots$$

$$= (b_{r-1}),$$

and hence $(b_0; b_1, b_2, \dots, b_{r-1} + 1, \bar{2}_{[k]}, \bar{1})$ is the principal convergent of the $(r - 1)^{th}$ order, while $(b_0; b_1, b_2, \dots, b_{r-1} + 1)$ is the first intermediate convergent between the principal convergents of the $(r - 2)^{th}$ and r^{th} orders.

We may now state the following results:

(i) The E_1 -transformation applied to the r^{th} partial fraction introduces after the principal convergent of the $(r - 2)^{th}$ order the first $(k + 1)$ intermediate convergents that come between the principal

convergents of the $(r-2)^{th}$ and r^{th} orders and then the principal convergents of the $(r-1)^{th}$ and r^{th} orders of the simple continued fraction.

(ii) The E_2 -transformation which applies only to the last term of a finite simple continued fraction, just misses out the penultimate convergent and interpolates before the last principal convergent the entire gamut of intermediate convergents.

The E_3 -transformation has a similar effect; but it preserves the next following principal convergent also.

(iii) The P-transformation applied to $\frac{1}{+2}$ (medial or final) simply substitutes an intermediate convergent for the preceding principal convergent and does not affect the other convergents. Thus, every one of the *new convergents* after the transformations is either a principal or intermediate convergent of the corresponding simple continued fraction.

N. B.—The E_1 -, E_2 -, and E_3 -transformations are called *extending transformations*, since they increase the number of convergents in the new continued fraction; while the P-transformations preserve this number.

1.2 Annihilatory Transformations

There are two fundamental transformations that may be employed with reference to consecutive unit partial quotients:

(1) Those which affect the partial quotient immediately preceding the first unit partial quotient;

(2) those which do not do so.

We call them ' C_1 ', ' C_2 ' respectively, when successive unit partial quotients are either modified or annihilated as noticed below.

1.21 The C_1 -transformation

We have

$$\begin{aligned} & (b_r, 1_{[t]}, b_{r+t+1}, \dots) \\ &= (b_r+1, \bar{2}, 1_{[t-2]}, b_{r+t+1}, \dots) \text{ by } (\beta) \\ &= (b_r+1, \bar{3}, \bar{2}, 1_{[t-4]}, b_{r+t+1}, \dots) \\ & \dots \end{aligned}$$

$$\begin{aligned}
 &= (b_r + 1, \overline{3} \left[\frac{t-2}{2} \right], \overline{2}, b_{r+t+1}, \dots) \text{ when } t \text{ is even } \dagger \\
 &\text{or } (b_r + 1, \overline{3} \left[\frac{t-1}{2} \right], \overline{b_{r+t+1} + 1}, \dots) \text{ when } t \text{ is odd } \dagger\dagger \left. \vphantom{\begin{aligned} &= (b_r + 1, \overline{3} \left[\frac{t-2}{2} \right], \overline{2}, b_{r+t+1}, \dots) \text{ when } t \text{ is even } \dagger \\ &\text{or } (b_r + 1, \overline{3} \left[\frac{t-1}{2} \right], \overline{b_{r+t+1} + 1}, \dots) \text{ when } t \text{ is odd } \dagger\dagger \end{aligned}} \right\} \dots (6)
 \end{aligned}$$

The expression $S_{r,n+n}$ is unaltered by this transformation; if $n = t + 2$ before the transformation, it is after the transformation equal to $\frac{t+4}{2}$ or $\frac{t+3}{2}$ according as t is even or odd. . . (7)

N. B.—This statement is true for the following transformation, only when t is even.

1.22 The C_2 -transformation

We write

$$\begin{aligned}
 &(b_r, 1_{[t]}, b_{r+t+1}, \dots) \\
 &= (b_r, 2, \overline{2}, 1_{[t-3]}, b_{r+t+1}, \dots) \\
 &= (b_r, 2, \overline{3}, \overline{2}, 1_{[t-5]}, b_{r+t+1}, \dots) \\
 &\dots \dots \dots \\
 &= (b_r, 2, \overline{3} \left[\frac{t-3}{2} \right], \overline{2}, b_{r+t+1}, \dots) \text{ when } t \text{ is odd } \dagger\dagger\dagger (> 1) \\
 &\text{or } (b_r, 2, \overline{3} \left[\frac{t-2}{2} \right], \overline{b_{r+t+1} + 1}, \dots) \text{ when } t \text{ is even } \left. \vphantom{\begin{aligned} &= (b_r, 2, \overline{3} \left[\frac{t-3}{2} \right], \overline{2}, b_{r+t+1}, \dots) \text{ when } t \text{ is odd } \dagger\dagger\dagger (> 1) \\ &\text{or } (b_r, 2, \overline{3} \left[\frac{t-2}{2} \right], \overline{b_{r+t+1} + 1}, \dots) \text{ when } t \text{ is even } \end{aligned}} \right\} (8)
 \end{aligned}$$

1.23 The $c_1 c_2^c$ -transformation

A third method of annihilation is what may be called the eclectic method, which is a combination of the C_1 -and C_2 -methods applied partially to the unit partial quotients. It is important to note that the partial C_1 -transformation applies only initially and absorbs an even number of consecutive unit partial quotients, while the partial C_2 -transformation applies initially, medially and finally, absorbing an odd number of consecutive unit partial quotients in the first two cases and an odd or even number in the last case, symbolised accordingly by c_2° or c_2^c . The eclectic transformation can, therefore, be only one of four types:

- | | |
|-------------------------------------|--------------------------------------|
| (i) $c_1 c_2 c_2 \dots c_2^c$; | (ii) $c_1 c_2 c_2 \dots c_2^\circ$; |
| (iii) $c_2^c c_2 c_2 \dots c_2^c$; | (iv) $c_2 c_2 c_2 \dots c_2^\circ$. |

† When $t=2$ the term $\overline{3}$ drops out; †† similarly when $t=1$ and ††† when $t=3$

Let t be the number of consecutive unit partial quotients affected by an eclectic transformation.

In the first two types, suppose c_1 absorbs k unit partial quotients and the c_2 's, (say r in number) absorb respectively $l_1, l_2, \dots, l_r$ quotients. Then the reduction in the number of terms due to this transformation is given by

$$\frac{k}{2} + \sum_{i=1}^{r-1} \frac{l_i - 1}{2} + \frac{l_r}{2} \text{ or } \frac{l_r - 1}{2} = \frac{t - r + 1}{2} \text{ or } \frac{t - r}{2} \quad \dots (9)$$

according as l_r is even or odd, since $t = k + \sum_{i=1}^r l_i$

The maximum reduction will occur when $r=1$ or 2, viz., for the types $c_1c_2^e$, $c_1c_2^o$, $c_1c_2c_2^e$, and $c_1c_2c_2^o$.

Similarly, it can be shown that the maximum reduction for the types (iii) and (iv) will occur when $r=2$ or 3, viz., for

$$c_2c_2^e, c_2c_2^o, c_2c_2c_2^e, \text{ and } c_2c_2c_2^o,$$

The scheme of reductions in the number of terms due to the application of the above eight special types of eclectic transformations is as follows:

Type	r	t	Reduction
$c_1c_2^e$	1	even	$t/2$
$c_1c_2c_2^o$	2	"	$(t-2)/2$
$c_2c_2^o$	"	"	"
$c_2c_2c_2^e$	3	"	"
$c_1c_2^o$	1	odd	$(t-1)/2$
$c_1c_2c_2^e$	2	"	"
$c_2c_2^e$	"	"	"
$c_2c_2c_2^o$	3	"	$(t-3)/2$

From the above tables we find that those which give the minimum number of partial quotients among the eclectic transformation are

(1) $c_1c_2^e$ when t is even,

and (2) $c_1c_2^o$, $c_1c_2c_2^e$, and $c_2c_2^e$ when t is odd, due to a reduction of $t/2$ terms in (1) and $\frac{t-1}{2}$ terms in (2).

But the C_1 -transformation gives a reduction of $t/2$ terms when t is even and $\frac{t+1}{2}$ terms when t is odd, while the C_2 -transformation gives the reduction of $t/2$ terms when t is even and $\frac{t-1}{2}$ terms when t is odd.

Hence, we may enunciate the result :

Of all the annihilatory transformations, the three transformations C_1 , C_2 , and $c_1c_2^t$ alone give the least number of terms when t is even, while C_1 is the only such transformation when t is odd. ... (10)

1.24 The Convergents of the annihilatory transformations

Applying (β) to the $(r+1)^{th}$ partial fraction of the simple continued fraction

$$(i) \quad (b_0; b_1, b_2, \dots, b_r, 1, b_{r+2}, 1, b_{r+4}, \dots \dots)$$

we get

$$(ii) \quad (b_0; b_1, b_2, \dots, b_r+1, b_{r+2}+1, 1, b_{r+4}, \dots \dots)$$

and note that the convergent of the r^{th} order in (ii) is the same as that of the $(r+1)^{th}$ order in (i), while all the preceding convergents in (ii) are the same as the corresponding ones in (i).

Applying (β) again to the $(r+2)^{th}$ partial fraction in (ii), we have

$$(iii) \quad (b_0; b_1, b_2, \dots, b_r+1, \overline{b_{r+2}+2}, \overline{b_{r+4}+1}, \dots \dots)$$

in which the convergent of the $(r+1)^{th}$ order is the same as that of the $(r+2)^{th}$ order in (ii) and that of $(r+3)^{th}$ order in (i).

Thus (β) applied to a unit partial fraction of (i) just annihilates only the preceding convergent, while it keeps intact all the others.

Now, the C_1 and C_2 -transformations are merely aggregates of (β) transformations applied to alternate unit partial fractions of a set, commencing from the first or the second fraction of the set, and therefore annihilate the series of alternate convergents commencing from that which just precedes the one corresponding to the initial unit partial fraction annihilated.

Similarly, the eclectic transformations also annihilate sets of alternate convergents in a manner appropriate to their eclectic nature.

Thus, the annihilatory transformations have the property of contracting or reducing the number of terms, when they are applied. They do not introduce any intermediate convergents like E_1 , E_2 , E_3 and P -transformations, while they annihilate some of the principal convergents of the simple continued fraction.

1.25 N. B.—From § 1.12 and § 1.24, we notice the important fact that the last two convergents of a finite $h. r. C. F.$ are always equal to the last two convergents of the corresponding $S.C.F.$ except when the

transformation E_2 (or P) has been applied to the last partial fraction of the S.C.F. (to convert it into the *h. r. C.F.* in question).

We call this the '*property of the last two convergents*' which holds even in the exceptional case, if we replace the last partial fraction $\frac{1}{-2}$ of the *h. r. C.F.* by $\frac{1}{-1} \frac{1}{+1}$.

1.3 We have now fully envisaged the three groups (E_1 , E_2 , E_3), (P), and (C_1 , C_2 , $c_1c_2 \dots c_2^c$, $c_1c_2 \dots c_2^o$, c_2 , $c_2 \dots c_2^e$, $c_2c_2 \dots c_2^o$) of transformations which convert a simple continued fraction into a half-regular one. We will next prove the converse, viz., that *any* half-regular continued fraction can be derived from the corresponding simple continued fraction by applying one or more of the above transformations in a non-overlapping manner. Some of the important types that thus occur will be studied in § 3.

1.4 Genesis of the half-regular continued fraction

Let $(b_0; b_1; b_2, \dots b_r, \overline{b_{r+1}, \dots b_s}, b_{s+1}, \dots)$

be a *h. r. C. F.*, which can be converted into the following C.F. by Lagrange's Rule ⁽²⁾, i.e., a repeated application of (α):

$$(b_0; b_1, \dots b_r - 1, 1, b_{r+1} - 2, \dots 1, b_s - 1, b_{s+1}, \dots)$$

where no element is negative but some may be zero†, as for example when $b_{r+1} = 2$. When one or more elements vanish, the C.F. admits of a condensation by the coalescence of the non-vanishing partial quotients before and after the zero; for obviously, $\frac{1}{a+} \frac{1}{0+} \frac{1}{b} = \frac{1}{a+b}$. Ultimately, after all the condensations are made, the result is a simple C.F. (S.C.F.)

If we now look at the process the other way, we easily find the transformations that have gone to alter the S.C.F. into the *h. r. C.F.* For example, when only two alternative elements have coalesced, either E_1 or P -transformation has been operative, while if successive alternate elements have coalesced E_1 , E_2 , or E_3 transformations must have taken effect. If there be no vanishing elements, evidently then one or more of the annihilatory transformations must have effected the change.

It will also be seen that these transformations have affected the partial quotients in order from left to right in such a non-

† No two zeros will occur consecutively but they may occur in alternate places.

overlapping manner that when one of the transformations has altered a certain number of positive partial fractions (of the S. C. F.) into negative ones, the next partial fraction to be affected by a fresh transformation is that which comes after the last negative partial fraction.

We may call this '*the principle of non-overlapping*'. An example will make this clear. Consider the finite h. r. C. F.

$$1 - \frac{1}{1 + \frac{1}{3 - \frac{1}{2 - \frac{1}{2 - \frac{1}{5}}}}}$$

which becomes after Lagrange's transformation

$$\begin{aligned} 0 + \frac{1}{1 + 0 + \frac{1}{2 + 1 + 0 + \frac{1}{1 + 0 + \frac{1}{1 + 0 + \frac{1}{1 + \frac{1}{4}}}}} \\ = \frac{1}{3 + \frac{1}{3 + \frac{1}{4}}} \quad (\text{S. C. F.}) \end{aligned}$$

In getting back from the S. C. F. to the h. r. C. F., the transformations to be employed from left to right are in order: E_1 , E_3 ; applied respectively to the first and second partial fractions (of the S. C. F.).

The types of transformation in any particular case are also easily suggested by the characteristic sequence of partial quotients associated with them; thus, in the notation of page 1, a solitary $\bar{1}$ indicates E_1 , a succession of $\bar{2}$'s followed by a partial quotient greater than 1 indicates E_3 , a succession of pure $\bar{3}$'s implies C_1 and so on.

It is now plain how the several groups of transformations discussed here are sufficient to generate a half-regular continued fraction from the corresponding simple continued fraction.

§ 2. Properties of Residues

2.1. Lemma I: Of two consecutive residues in a h. r. C. F.† in which the first is led by a unit partial fraction, either the absolute value of one is less and that of the other greater, than $\frac{1}{2}(\sqrt{5}-1)$ or both are absolutely equal to $\frac{1}{2}(\sqrt{5}-1)$. When two consecutive residues have their absolute values equal to $\frac{1}{2}(\sqrt{5}-1)$, the first of these is led by a unit partial fraction.

The proof is immediate, since

$$\frac{1}{1 + \frac{1}{x}} \leq \frac{\sqrt{5}-1}{2} \text{ according as } \frac{1}{x} \geq \frac{\sqrt{5}-1}{2}$$

† This proposition is also true for the S. C. F. though certain qualifications are superfluous in this case

Again, if $|\xi|=1$, $\frac{1}{a} + \frac{\xi}{b} = \frac{\sqrt{5}-1}{2}$, and $\frac{1}{b} = \frac{\sqrt{5}-1}{2}$

we have $a = \frac{\sqrt{5}+1}{2} - \xi$, $\frac{\sqrt{5}-1}{2}$, and equating the rational and irrational parts, we find $a=1$, $\xi=1$.

Lemma II: If three consecutive partial fractions in a *h. r. C. F.* occur as in $\frac{\pm 1}{2} - \frac{1}{2} \pm a \pm \dots$ then the absolute values of the residues led by them are not all less than $\frac{1}{2}(\sqrt{5}-1)$.

For, $\frac{1}{2} - \frac{1}{y} \leq \frac{1}{2}(\sqrt{5}-1)$ according as $y \geq 2 + \frac{1}{2}(\sqrt{5}-1)$; and if $y = 2 + \frac{1}{a} + \dots$, and $\frac{1}{2} - \frac{1}{y} < \frac{1}{2}(\sqrt{5}-1)$, then $\frac{1}{a} + \dots > \frac{1}{2}(\sqrt{5}-1)$, while if $y = 2 - \frac{1}{a} + \dots$, then $y < 2 + \frac{1}{2}(\sqrt{5}-1)$ and $\frac{1}{2} - \frac{1}{2 - \frac{1}{a} + \dots}$ is greater than $\frac{1}{2}(\sqrt{5}-1)$.

Cor: $\frac{1}{2} - \frac{1}{2 - \frac{1}{a} \pm \dots}$ and $\frac{1}{2} - \frac{1}{1 + \frac{1}{a} \pm \dots}$ are both greater than $\frac{1}{2}(\sqrt{5}-1)$ when $a \pm \dots > 1$.

2.2 The E_1 -transformation introduces always one residue of the type $\frac{1}{1+a}$ or $\frac{1}{2-1+a}$, while E_2 and E_3 -transformations always introduce one of the type $\frac{1}{2-} - \frac{1}{2-} - \frac{1}{a}$. Hence, a residue absolutely equal to or greater than $\frac{1}{2}(\sqrt{5}-1)$ occurs necessarily on account of any of the extending transformations.

2.3. The C_1 -transformation, partial or otherwise, introduces residues of the types $\frac{1}{\pm 3-}$ or $\frac{1}{-2+}$, which are both numerically less than $\frac{1}{2}$ and therefore less than $\frac{1}{2}(\sqrt{5}-1)$.

The C_2 -transformation, partial or otherwise, introduces also residues less than $\frac{1}{2}$ except in the case of the first residue, which is led by $\frac{1}{2-}$. This first residue is equal to (1_∞) or $(1_{[r]}, \dots)$ where r is the number of unit partial quotients absorbed by this C_2 -transformation and all the following ones (if any) in the particular eclectic transformation considered.

Now $(1_{[r]}, a) < (1_\infty) = \frac{1}{2}(\sqrt{5}-1)$ according as r is odd or even, when $a \geq 2$. Obviously, r is odd for at least one c_2 -transformation

in any eclectic transformation which either involves more than one partial c_2 -transformation or is of the type $c_1c_2^\circ$, the number of unit partial quotients absorbed by c_2° or $c_2c_2^\circ$ being always odd; and r is certainly even for the transformation $c_1c_2^\circ$ as well as the complete C_2 -transformation absorbing an even number of unit partial fractions.

Hence, a C_2 -transformation, partial or otherwise, introduces one or more residues equal to or greater than $\frac{1}{2}(\sqrt{5}-1)$ in all cases except where it occurs once and once only and absorbs a finite and even number of unit partial quotients. It is therefore remarkable that those annihilatory transformations which give the least number of partial quotients are also just those which make the residues absolutely less than $\frac{1}{2}(\sqrt{5}-1)$ (11)

2.4 As regards the medial P-transformation, it introduces (vide Lemma II) a residue, absolutely equal to or greater than $\frac{1}{2}(\sqrt{5}-1)$ only under the following circumstances:

(i) When the partial $\frac{1}{+2}$ to which it is applied, immediately succeeds a unit partial fraction, and

(ii) When the residue in the S.C.F. succeeding $\frac{1}{+2}$ is either equal to $\frac{1}{2}(\sqrt{5}-1)$ or in the form $(t_{[r]}, a, \dots)$ where $a \geq 2$ and r odd. (12)

Hence, in order that a P-transformation may secure a residue absolutely less than $\frac{1}{2}(\sqrt{5}-1)$ in a *h. r. C.F.*, the unit partial fractions (if any) just preceding $\frac{1}{+2}$ must first be dealt with by appropriate annihilatory transformations and then the P-transformation may be applied to $\frac{1}{+2}$ provided it survives the antecedent annihilatory transformations and is not followed immediately by an odd number or an infinite number of unit partial quotients. Indeed, the partial fraction $\frac{1}{+2}$ will not survive, if it is either immediately followed by an infinite or odd number of consecutive unit partial fractions or immediately preceded by unit partial fractions, when these are so annihilated that the residues are absolutely less than $\frac{1}{2}(\sqrt{5}-1)$, unless the immediately preceding unit partial fractions (if any) are even in number and are operated upon by the C_1 -transformation. When the batch of unit partial fractions immediately following $\frac{1}{+2}$ is even, the application of the P-transformation first and then the C_1 -transformation is

tantamount to applying first the C_2 -transformation to the even number of unit partial fractions and then applying the P-transformation to $\frac{1}{+2}$ which will not have been affected by the C_2 -transformation. Thus, there is nothing lost by applying the P-transformations after the annihilatory transformations $C_1, C_3, c_1c_2^c$ have taken effect appropriately, in the process of conversion of a S.C.F. into a h.r. C.F. with all residues absolutely less than $\frac{1}{2}(\sqrt{5}-1)$.

2.5 Summing up, we note that the extending transformations always introduce a residue numerically equal to or greater than $\frac{1}{2}(\sqrt{5}-1)$, while among the annihilatory transformations, only $C_2, c_1c_2^c$ applied to a batch of even number of consecutive unit partial fractions and C_1 applied to any batch (finite or infinite) of such fractions introduce residues all absolutely less than $\frac{1}{2}(\sqrt{5}-1)$; but the P-transformation will introduce residues absolutely equal to or greater than $\frac{1}{2}(\sqrt{5}-1)$, only when the partial fraction to which it is applied comes immediately after a unit partial fraction or before a batch of an odd number or infinite number of such fractions. In respect of the combination of the P and $C_1, C_2, c_1c_2^c$ transformations securing the absolute values of all residues less than $\frac{1}{2}(\sqrt{5}-1)$, it will be convenient to apply the appropriate annihilatory transformations first and then the P-transformations optionally to the surviving partial fractions of the type $\frac{1}{+2+}$ preceded and followed by partial quotients greater than 1, without violating 'the principle of non-overlapping' observed in § 1.4 *supra*.

§ 3. Varieties

3.1 Vahlen's Theorem⁽³⁾: The reduced fraction p/q ($q > 0$) admits of exactly q different ways of development as a finite half-regular continued fraction.

Let p/q be expressed as the S.C.F.

$$F_n \equiv (b_0; b_1, b_2, \dots, b_n)$$

where $b_n \geq 1$; and let B_r be the number of ways in which $F_r (\equiv (b_0; b_1, \dots, b_r))$ can be developed into finite h. r. C.F.'s.

The required h. r. C.F. developments of F_n (including F_n) can be divided into two classes:

I. Those which involve one of our transformations (β, E_1, E_2, E_3, P , and the identical transformation) affecting the last partial fraction to which it is not directly applied; in other words, the

transformation is applied to the penultimate partial fraction so as to affect the last partial fraction.

II. Those which involve a transformation applied to the last partial fraction. The processes by which these two classes are respectively derived will be referred to as the *first* process and the *second* process.

If $b_n > 1$, we may apply to $\frac{1}{+b_{n-1}}$

- (i) E_3 -transformation when $b_{n-1} > 2$, or
- (ii) P- " " $b_{n-1} = 2$, or
- (iii) β - " " $b_{n-1} = 1$

these three being the only transformations which affect also b_n .

With every one of the B_{n-2} h.r. C.F's equal to F_{n-2} , modified if necessary, so as to possess 'the property of the last two convergents' (vide § 1.25), we may associate one of the above transformations and derive the same number of h.r. C.F's equal to F_n .

This is the *first* process.

Next, if $b_n > 2$, we may apply to $\frac{1}{+b_n}$ the identical transformation (i.e. keep it intact) or the E_1 -transformation in one of $b_n - 2$ ways, or the E_2 -transformation and associate each of these b_n different ways on the whole with each of B_{n-1} h.r. C.F's equal to F_{n-1} modified as before, so as to obtain $b_n B_{n-1}$ different h.r. C.F's equal to F_n .

When $b_n = 2$, we may apply to $\frac{1}{+2}$ either the P-transformation or the identical transformation and so deal with it, again, in $b_n (= 2)$ ways and by association obtain as before $b_n B_{n-1}$ different h.r. C.F's equal to F_n .

This is the *second* process.

On the whole, from the two processes together, we can derive $B_{n-2} + b_n B_{n-1}$ different h.r. C.F's equal to F_n .

When $b_n = 1$, we can either add 1 to the last partial quotient of each of the B_{n-1} h.r. C.F's equal to F_{n-1} , modified as aforesaid where necessary or associate the partial fraction $\frac{1}{+(b_{n-1} + 1)}$ with each of the B_{n-2} h.r. C.F's equal to F_{n-2} , modified as before. We get thus $B_{n-2} + b_n B_{n-1}$ different h.r. C.F's again as before.

The associations mentioned here are valid on account of 'the property of the last two convergents'. For, if

$$(b_0; b_1, b_2, \dots, b_r) = c_0 + \frac{\xi_1}{c_1} + \frac{\xi_2}{c_2} + \dots + \frac{\xi_k}{c_k} \text{ (where } |\xi_{k'}| = 1, k' = 1, 2, \dots, k)$$

such that the last two convergents of the two continued fractions are equal each to each, then,

$$(b_0; b_1, b_2, \dots, b_r, b_{r+1}, \dots) = c_0 + \frac{\xi_1}{c_1} + \frac{\xi_2}{c_2} + \dots + \frac{\xi_k}{c_k} + \frac{1}{b_{r+1} + \dots}$$

for the convergent of the $(r+t)^{\text{th}}$ order on the *l.h.s.* is equal to that of the $(k+t)^{\text{th}}$ order on the *r.h.s.*, for $t=1, 2, \dots$

We may now state the result:

If $F_r = (b_0; b_1, b_2, \dots, b_r)$ and B_r the number of *h.r.* C.F.'s equal to F_r , then $B_n = B_{n-2} + b_n B_{n-1}$ (13)

Obviously, $B_0 = 1$, and $B_1 = b_1$ (vide 'Second' process).

Hence $B_n = K(1, n)^{(4)} = q$ (in the notation of continuants)

This proves Vahlen's theorem.

Note: Though this proof is longer than the usual one (vide Perron, p. 157), it has the advantage of giving a better insight into the structure of the *h.r.* C.F.'s and their enumeration.

3.2 We will next discuss the relative lengths of the *h.r.* C.F.'s equal to $F_n = (b_0 + \frac{1}{b_1} + \dots + \frac{1}{b_n})$, ($b_n > 1$), the length being defined by the number of partial fractions in a continued fraction.

Let the partial fractions of F_n to which one or more of the transformations of § 1 may be applied to convert it into a *h.r.* C.F. be in order

$$\frac{1}{+b_{i_1}}, \frac{1}{+b_{i_2}}, \dots, \frac{1}{+b_{i_k}}$$

where $i_1 < i_2 < \dots < i_k \leq n$

When $b_{i_r} = 1$ ($r \leq k$), the β -transformation just annihilates it.

When $b_{i_r} > 1$, the transformations E_2, E_3, P (wherever they are applicable) replace the term by exactly $(b_{i_r} - 1)$ terms; while the E_1 -transformation replaces it by not more than b_{i_r} or $b_{i_r} - 1$ terms according as it is applied medially or finally.

Hence, the total effect of the transformations is such that the number of partial fractions in a h. r. C. F development can never exceed

$$n - k + \sum_{k'=1}^k b_{i_{k'}} \text{ or } n - k - 1 + \sum_{k=1}^k b_{i_k}$$

according as $i_k < n$ or $i_k = n$

Since $b_n > 1$, $\sum_1^n b_r > n - k + b_{i_1} + b_{i_2} + \dots + b_{i_k}$ if $i_k = n$; and

$$\sum_1^n b_r > n - k + b_{i_1} + b_{i_2} + \dots + b_{i_k} \text{ if } i_k < n \\ > n - k + 1 + b_{i_1} + b_{i_2} + \dots + b_{i_k} .$$

In all cases, therefore, the upper limit for the number of partial fractions in a finite h. r. C. F development of F_n is

$$b_1 + b_2 + \dots + b_n - 1 = S_{1,n} - 1 \quad \dots (14)$$

3.21. From (14) we infer that the longest finite h. r. C. F development of F_n cannot have more than $S_{1,n} - 1$ partial fractions. We will now prove that there *two* and only *two* longest h. r. C. F's having exactly $S_{1,n} - 1$ partial fractions.

To attain the maximum number, the only applicable transformations are E_1 and E_2 (or P). For, the E_3 -transformation, applied, say to $\frac{1}{b_{i_{k'}}} (k' \leq k)$ replaces it and the next partial fraction by just $b_{i_{k'}} (< b_{i_k} + b_{i_{k'+1}})$ terms; while the P-transformation as well as the annihilatory transformation applied to the same replaces it by less than $b_{i_{k'}}$ terms, when $b_{i_{k'}} = 2$ or 1. Further, the extreme E_1 -transformation applied *medially* gives rise to just as many new partial fractions as the units in the partial quotient replaced, while it gives *finally* just one less than this number in the same way as the E_2 (or P) transformation. The two longest (in the sense, not shorter than any other) h. r. C. F's are therefore obtained by applying the extreme E_1 -transformation to all the partial fractions except the last (barring unit partial fractions) and transforming the last either by the extreme E_1 -transformation or the E_2 (or P) transformation. In each of these two cases the number of partial fractions is the maximum possible, viz. $b_1 + b_2 + \dots + b_n - 1$ and the residues are all numerically greater than $\frac{1}{2}$ except the last which is numerically equal to $\frac{1}{2}$.

Further, the difference between the number of the partial quotients of either of the longest h. r. C. F's and their sums is either

equal to or one more than the corresponding difference of the S. C. F₁, according as the E₂ (or P) transformation is not or is employed.

3.3. The reduced half-regular continued fraction (reduziert regelmässigen Kettenbrüchen) ... (1)

A reduced *h. r. C. F* is one in which all the partial fractions are negative⁽⁰⁾; this may be briefly referred to as *r. r. C. F*.

To convert $F_n = (b_0; b_1, b_2, \dots, b_n)$ a S. C. F into a *r. r. C. F* we employ the E₃ (or P or β) transformation with respect to the alternate terms

$$+b_1, +\frac{1}{b_3}, \dots \text{ and } -\frac{1}{b_{n-2}} \text{ or } -\frac{1}{b_{n-1}}$$

according as n is odd or even and further apply the E₂ (or P) transformation to the last partial fraction only when n is odd. For example, $(b_0; b_1, b_2, \dots, b_{2n})$

$$= (b_0 + 1; \bar{2}_{[b_1-1]}, \overline{b_2 + 2}, \dots, \bar{2}_{[b_{2n-1}-1]}, \overline{b_{2n} + 1}). \quad \dots (1)$$

$$(b_0; b_1, b_2, \dots, b_{2n+1})$$

$$= (b_0 + 1; \bar{2}_{[b_1-1]}, \overline{b_2 + 2}, \dots, \bar{2}_{[b_{2n-1}-1]}, \overline{b_{2n} + 2}, \bar{2}_{[b_{2n+1}-1]}). \quad \dots (2)$$

If the S. C. F. be infinite, the corresponding *r.r. C.F* is

$$(b_0 + 1; \bar{2}_{[b_1-1]}, \overline{b_2 + 2}, \bar{2}_{[b_3-1]}, \overline{b_4 + 2}, \dots \text{ ad inf.}) \quad \dots (3)$$

With the help of examples (1) and (2), it is easy to verify the following:

If (S, N) , (S', N') represent within their brackets the sum and the number of partial quotients of a finite simple continued fraction and the corresponding reduced regular continued fraction then (i) $S' - S = N' - 1$, whether N is odd or even; and (ii) the sum of the partial quotients of the S. C. F. in the even places is equal to N' or $N' - 1$ according as N is even or odd. ... (15)

N. B.— Even if the first partial quotient is 0, it must be counted as one of the partial quotients for the purpose of the above result.

3.4 The Shortest half-regular continued fractions.

The shortest *h.r. C.F.* development of a given rational number cannot contain partial fraction groups of the following types:

$$(1) \frac{1}{\pm 1}; (2) \pm \frac{1}{a} + \frac{1}{2} - \frac{1}{2} \pm \frac{1}{b}; (3) \pm \frac{1}{a} - \frac{1}{2} - \frac{1}{2} \pm \frac{1}{b}.$$

For, all these are capable of reduction to a lesser number of terms. In fact (1) can be annihilated by (β) , (2) can be reduced to $\frac{1}{(a+1)} - \frac{1}{3} + \frac{1}{(b+1)}$, and (3) to $\frac{1}{(a-1)} + \frac{1}{3} + \frac{1}{(b+1)}$ (the upper signs going together and similarly the lower ones).

The transformations E_1, E_2, E_3 always introduce partial fractions of the above kind, especially types (1) and (3) and therefore cannot give rise to the shortest *h.r.* C.F. development.

As regards unit partial fractions, we have seen that the maximum reduction in their number is achieved by the transformations $C_1, C_2, c_1c_2c_3$ appropriately applied according to (10). With reference to the P-transformation which does not affect the number of terms, it may be noticed that if it is applied to $\frac{1}{+2}$ adjoining unit partial fractions (if any) before the transformation C_1, C_2 , or $c_1c_2c_3$ has taken effect according to (10), the result will be to reduce the number of adjoining unit partial fractions by 1 or 2 and thereby modify the maximum reduction possible in the number of these unit partial fractions according to the following scheme:

Number of unit partial fractions		Maximum reductions possible as per (10).	
Before	After	Before	After
the P-transformation		the P-transformation	
$2n$	$2n-1$	n	n
$2n$	$2n-2$	n	$n-1$
$2n+1$	$2n$	$n+1$	n
$2n+1$	$2n-1$	$n+1$	n

Thus the effect of the premature P-transformation is not to reduce the S. C. F. to the shortest *h.r.* C.F. except when it applies just to only one partial fraction $\frac{1}{+2}$ adjoining an even batch of unit partial fractions. In the exceptional case, it is easily seen that the effect of applying the P-transformation first and then the C_1 -transformation is the same as applying first a C_1 or C_2 transformation and then the P-transformation according as $\frac{1}{+2}$ adjoins the batch of unit partial fractions on the right or on the left. Hence, to obtain the shortest *h.r.* C.F. development, we may well postpone the P-transformation which is optional until all the appropriate annihilatory transformations have taken place so as to effect maximum reduction in the number of terms. As this process is not generally unique,

there are often several half-regular C. F. developments of the same length but shorter than others, for a given finite S. C. F.

Provided the P-transformations be nowhere applied in the above process the sum of the partial quotients as well as their number is the same for all such shortest C. F. developments, while the sum of these two quantities is equal to the corresponding sum for the simple continued fraction. (Vide (7)). ... (16)

In this connection, we may note that the number of ways in which an even number (say $2n$) of successive unit partial quotients occurring between two partial quotients greater than 1 can be reduced to the minimum number n in $(n+1)$ ways, which comprise $C_1 -$, $C_2 -$, and $(n-1)$, $C_1 C_2$ transformation

$$\text{Example: } 2 + \frac{1}{1 + \frac{1}{1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \frac{1}{1 + 2}}}}}}}}}$$

can be transformed into one of the shortest *h.r.* C.F.'s without applying the P-transformations, 3×2 or 6 ways, for the two batches of four and two unit partial fractions can be modified respectively in 3 ways and 2 ways which may be associated one with the another, while the odd batch can be modified only in one way i.e. by applying the C_1 -transformation.

The total number of ways of obtaining the shortest *h.r.* C.F.'s equal to a given S. C. F. is discussed by Tietze⁽⁵⁾ and the discussion need not be reproduced here.

3-41 Among the equi-long shortest *h.r.* C.F. developments *two* are well known :

- (1) the continued fraction to the nearest integer (nach nächsten . Ganzen) in which the residues are never absolutely greater than $\frac{1}{2}$;
- (2) the singular continued fraction (due to Hurwitz).

The first of these is obtained by applying the C_1 -transformation uniformly to all the batches of unit partial fractions and the *h.r.* C.F. so obtained, may be represented by

$$\left(c_0; \frac{\xi_n}{c_n} \right)_{n=1}^{n=k},$$

which is characterised by the properties

- (i) $c_n > 1$ ($n \leq k$) and (ii) $c_n + \xi_{n+1} > 1$ ($n < k$),
- easily verified by (6).

The singular continued fraction is derived by employing the C_1 -transformation for odd batches and the C_2 -transformation for even batches of unit partial fractions. It is characterised by the properties (i) $c_n > 1$ and $c_n + \varepsilon_n > 1$ ($n \leq k$) verified by (6) and (8).

If the continued fractions be infinite, the restrictions for n have to be removed.

3.42 From § 2.5, it will be clear that the transformations used in converting a simple continued fraction into one of the shortest half-regular continued fractions are the same as those which secure the residues absolutely less than $\frac{1}{2}(\sqrt{5}-1)$. Hence, *all h.r. C.F. developments of a rational number of the same length as the corresponding C.F. to the nearest integer, are characterised by the property that its absolute residues are less than $\frac{1}{2}(\sqrt{5}-1)$. (Tietze)* ... (17)

Perron calls such continued fractions 'kanonisch'.

3.5 The number of ways of applying the annihilatory transformations to a batch of t consecutive unit partial quotients preceded and followed by partial quotients greater than 1, (occurring in a S. C. F.) will now be discussed

The annihilatory transformations operate in one of *four* following ways:

- (i) by affecting the t unit partial quotients only;
- (ii) „ $(t+1)$ consecutive partial quotients commencing from the one just preceding the first unit partial quotient:
- (iii) „ $(t+1)$ consecutive partial quotients ending with the one just following the last unit partial quotient;
- and (iv) „ $(t+2)$ consecutive partial quotients including the two partial quotients adjoining on either side the batch of t unit partial quotients.

The total number of ways of annihilation accordingly corresponds to the sum of the number of ordered partitions of

- (1) t , (2) $t+1$ taken twice (3) $t+2$

into a certain number of 3's and 2's, the first part being always 3. This is easily seen from the fact that the effect of the transformations C_1 or C_2 (partial or otherwise) is to modify the first three consecutive terms and the succeeding terms in pairs.

§ 4. 'Raschesten Kettenbruch entwicklung'

4.1 Following ^(*) Tietze's ideas, we introduce the terms 'more rapidly' 'more slowly', 'equally rapidly' converging in respect of two half-regular continued fraction developments of the same irrational number, say ω , whose convergents are respectively:

$$(1) \quad P_1, P_2, \dots, P_n, \dots$$

$$(2) \quad Q_1, Q_2, \dots, Q_n, \dots$$

by the definition:

The first continued fraction is more rapidly (or more slowly) converging than the second, when for all positive integral n greater than a certain index m (a positive integer),

$$|P_n - \omega| < (\text{or } >) |Q_n - \omega|;$$

otherwise, they are said to be equally rapidly convergent.

Let the S. C. F. development of ' ω ' be

$$(a_0; a_1, a_2, \dots); \quad \dots \quad (a)$$

let the *h. r. C. F* development obtained by using the extreme medial E_1 -transformations to (a) be

$$\left(b_0; \frac{E_1}{b_1}\right)_1^\infty; \quad \dots \quad (b)$$

and that due to the transformation c_1 applied to all batches of unit partial fractions in (a) be

$$\left(c_0; \frac{E_1}{c_1}\right)_1^\infty; \quad \dots \quad (c)$$

and let $\left(d_0; \frac{E_1}{d_1}\right)_1^\infty$, be any other *h. r. C. F* development of ω . (d)

Tietze has shown that (b) is the most slowly convergent and (c) one of the most rapidly convergent C. F.'s (raschesten Kettenbruch-entwicklung).

4.2. With a slight modification of Tietze's definition, we define the most rapidly convergent continued fraction developments of an irrational number ω by the following property:

If an infinite h. r. C. F, say (d), development of an irrational number ω has infinitely many convergents of the same order and value as the h. r. C. F, (c), development to the nearest integer, then (d) is one of the 'raschesten' variety.

Let P_n, Q_n be the convergents of the n^{th} order belonging to (d) and (c) respectively. If $P_n = Q_n$, it is evident that they correspond to

$$\left(d_0; \frac{\xi_v}{d_v}\right)_1^n \text{ and } \left(c_0; \frac{\xi_v}{c_v}\right)_1^n$$

which are both shortest *h. r. C. F.* developments, by § 3.4, of the same rational number; the first of these is therefore capable of being derived from the *S. C. F.*, say, $F_t \equiv (a_0; a_1, a_2, \dots, a_t)$ by means of the *P*-transformation and annihilatory transformations; further P_n, Q_n are both principal convergents of (a) .

Suppose there are infinitely many n 's for which $P_n = Q_n$, then the transformations by which (d) is derived from (a) must consist only of the *P*- and (β) -transformations.

$$\text{If, now, } \left(d'_0; \frac{\xi_v}{d'_v}\right)_1^\infty \quad \quad (d')$$

be the *h. r. C. F.* development derived from (a) by applying only the annihilatory transformations without the *P*-transformations, that are employed in obtaining (d) , then the convergents of the n^{th} order pertaining to (d) and (d') will coincide whenever they are both principal convergents of (a) .

But, every convergent of (d') is a principal convergent of (a) ; so that if P'_n be the (n^{th}) convergent of (d') and $P_n = Q_n$, then $P'_n = P_n = Q_n$.

We have infinitely many such n 's. Hence (d') is also of the 'raschesten' variety, by definition.

4.3. We can easily show that the annihilatory transformations employed in deriving (d') from (a) are just of the type c_1, c_2 , or $c_1 c_2'$ applied to even batches of consecutive unit partial quotients and c_1 to other batches, including, if any, the infinite one. For, otherwise, by taking n sufficiently large, we will have the shortest continued fraction development $\left(d'_0; \frac{\xi_v}{d'_v}\right)_1^n$ obtained by applying at least to one batch of successive unit partial fractions of the *S. C. F.*, F_t say, an eclectic transformation different from the appropriate one; which is impossible by (10). We may note in this connection, that, if $a_t = 1$, it is either the last partial quotient of a batch of an even number of unit partial quotients in (a) or a medial partial quotient occupying an odd rank in a batch of unit partial quotients. If (a) should end in an infinite number of unit partial fractions which are modified in

(d') by a transformation other than c_1 , then we may take n sufficiently large so that the shortest *h. r. C. F.* development $\left(d'_0; \frac{\xi_v}{d'_v}\right)_1^n$ is obtained from F_i where $a_i=1$ by annihilatory transformations the last of which is different from C_1 and therefore not appropriate to the last batch of unit partial fractions in F_i , which is necessarily odd. This is again a contradiction to (11).

Again, by (11), the residues of (d') are all absolutely less than $\frac{1}{2}(\sqrt{5}-1)$; and by Lemma II, a P-transformation applied to $\frac{1}{+2}$ which is followed by a positive residue less than $\frac{1}{2}(\sqrt{5}-1)$ does not introduce any residue absolutely greater than or equal to $\frac{1}{2}(\sqrt{5}-1)$. Hence the residues of (d) are also all absolutely less than $\frac{1}{2}(\sqrt{5}-1)$. This proves Tietze's⁽⁵⁾ famous theorem, which may be quoted here in the original:

'Die raschesten halbregelmässigen Entwicklung einer Zahl sind jene halbregelmässigen Entwicklung, bei denen jeder Entwicklungsrest r_{v+1} der Bedingung genügt

$$|r_{v+1}| < \frac{1}{2}(\sqrt{5}-1) = 0.6180339887 \dots (r=0, 1, \dots)$$

Tietze's theorem itself follows from the observations in § 2.5 and the properties of convergents mentioned in § 1.12 (iii) and § 1.24.

We may be permitted to conclude here with the remark that we have been able to give a simple and unified view of the half-regular continued fraction by a detailed examination of the properties of the two fundamental transformations (α) and (β) and the *ten* derived ones,

viz. $E_1, E_2, E_3, P, C_1, C_2, C_1C_2, \dots, C_2', C_1C_2' \dots, C_2^0,$
 $C_3C_2' \dots, C_2',$ and $C_2C_2' \dots, C_2^0.$

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ON WONDERFUL DEMLO NUMBERS

BY

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1. The object of this paper is to discuss the rationale of certain curious results in the multiplication of numbers written in the usual scale of ten, an example of which is the following :—

$$1234567654321 \times 243 = 2999999940000003.$$

I call this a “*partition and insertion property*” since, when the multiplier 243 is partitioned into digits and repeated digits 9 and 0 are inserted we get the product. I show in this paper how to obtain in other numbers with this property.

Notation and Definitions

2. To save space I shall write $(a)^n$ to indicate the digit a , repeated n times. Thus $(1)^2 0 2(9)^3 = 1102999$.

I use the word “Demlo Number” to denote an integer which may be partitioned into three groups of which the middle group is of the form $(a)^l$ and the sum of the numbers in the first and last groups (supposed to have the *same* number of digits) is of the form $(a)^m$. Thus 12488764 is a Demlo Number of which the middle part is 88 and whose extreme groups 124 and 764 add up to 888.

A special class of Demlo Numbers are “Wonderful Demlo Numbers” which are defined as follows :—

Def: A wonderful demlo number is the square of an integer of the form $(1)^l$ where $1 < l \leq 9$. Thus

$$121; 12321; 1234321; \text{ etc. and } \{1^{(9)}\}^2 = 13345678987654321$$

are all wonderful demlo numbers. Here the middle part consists of a single digit while the first and last groups consist of numbers in rising or falling order, I shall use W to denote a wonderful demlo number.

Def: A Partition number is an integer of the form $9 \times ab$ where $b \geq (a+1)$ and $a+b \leq 9$. I shall use the letter P to denote a “partition number.”

Example :— 9×45 is a partition number.

The Partition and insertion property

3. LEMMA I: If ABC be the successive digits of a partition number, then $A = a$, $B = b - a - 1$, $C = 10 - b$.

$$\text{For } ABC = ab \times 9 = ab(10 - 1) = ab0 - ab = a(b - a - 1)(10 - b).$$

$$\text{Thus } 35 \times 9 = 315 = 3(5 - 3 - 1)(10 - 5).$$

LEMMA II: If W be the square of $(1)^t$, then

$$W \times 9 = (1)^t (0)^t - (1)^t.$$

$$\begin{aligned} \text{For, } W \times 9 &= \{(1)^t\}^2 \times 9 = \left\{ \frac{(10)^t - 1}{9} \right\}^2 \times 9 \\ &= \frac{(10)^{2t} - 2(10)^t + 1}{9} \\ &= (10)^t \left\{ \frac{(10)^t - 1}{9} \right\} - \frac{(10)^t - 1}{9} = (1)^t (0)^t - (1)^t. \end{aligned}$$

$$\text{Example: } 12321 \times 9 = 111000 - 111.$$

$$\text{THEOREM I: } ABC \times W = A(c)^{t-1}B(9-c)^{t-1}C$$

$$\text{where } P = ABC \text{ and } c = a + b.$$

$$\begin{aligned} \text{For } W \times ABC &= W \times P = W \times 9 \times ab \\ &= \{(1)^t (0)^t - (1)^t\} \times ab \text{ by Lemma II} \\ &= a(c)^{t-1} b(0)^t - a(c)^{t-1}b \\ &= a(c)^{t-1} (b - a - 1) (9 - c)^{t-1} (10 - b) \\ &= A(c)^{t-1}B(9 - c)^{t-1}C \text{ from Lemma I} \end{aligned}$$

Thus $1234567654321 \times 243 = 2999999940000003$ (Here $243 = 9 \times 27$, so $c = a + b = 2 + 7 = 9$ and $9 - c = 0$ and both these are repeated.)

$$\text{Corollary I: } \{(3)^t\}^2 = (1)^{t-1} 0(8)^{t-1} 9$$

$$\text{For } \{(3)^t\}^2 = \{(1)^t\}^2 \times 9 = W \times P.$$

Thus the square of 3333 is 11108889.

$$\text{Corollary II: } (3)^t (6)^t = (2)^{t-1} 1 (7)^{t-1} 8.$$

$$\text{Corollary III: } \{(6)^t\}^2 = (4)^{t-1} 3 (5)^{t-1} 6.$$

$$\text{Corollary IV: } \{(9)^t\}^2 = (9)^{t-1} 8 (0)^{t-1} 1.$$

THEOREM III: There are only 25 partition numbers P altogether and they are: 9, 18, 27, 36, 45, 54, 63, 72, 81, 108, 117, 126, 135, 144, 153, 162, 207, 216, 225, 234, 243, 306, 315, 324 and 405.

For, from the definition $P=9 \times ab$, where $a+b=9$ and $b \geq a+1$
Hence a and b can assume only the following values :

a	b
0	1, 2, 3, 4, 5, 6, 7, 8, 9
1	2, 3, 4, 5, 6, 7, 8
2	3, 4, 5, 6, 7
3	4, 5, 6
4	5

Examples to Illustrate Theorem III

$$123454321 \times 216 = 26666133336$$

Here the digits 2, 1, 6 are partitioned and 3 and 6 are inserted in the gaps. Similarly

$$123454321 \times 315 = 38888111115$$

Again

$$12345654321 \times 108 = 1333330666668.$$

NOTES AND DISCUSSIONS

Progress in the History of Mathematics During the Last Forty Years

In 1898 the first number of volume 1, entitled "Grundlagen der Arithmetik" of the now widely used *Encyklopädie der mathematischen Wissenschaften* was published. On page 12 thereof there appears a note (18) on the negative numbers which involves various statements which are now commonly known to be incorrect and hence they exhibit decided progress in the history of our subject during the last forty years as it may be assumed that such a standard work in its day represents about the best that was then available along the lines treated. One of these statements is that the first traces of negative numbers appear in the work of the Indian mathematician Bhaskara (born in 1114) who distinguished between the positive and the negative value of the square roots of a positive number. On the contrary, it is now known that negative numbers were used much earlier in India and that the writings of the ancient Babylonians already contain such common rules in multiplication as $+ - = - + = -$, and $- - = +$. Cf. *Zentralblatt für Mathematik*, volume 16, page 385 (1937).

The noted incorrect assertion in regard to the first traces of negative numbers is followed in the same note by the assertion that the Arabs recognized negative roots of an equation which is equally commonly known now to be incorrect. In particular, J. Tropicke observes in his widely known *Geschichte der Elementar-Mathematik*, volume 2, page 97 (1933), that negative solutions were not allowed by the Arabians as may be seen, among other places, in the text-book on algebra by Alkarhi, about 1010. Minor misstatements in the same note are that the *Ars Magna* of Cardan appeared in 1550 instead of in 1545, and that T. Harriot, about 1600, considered negative numbers by themselves and allowed them to constitute a member of an equation. It is now known that the Babylonians already allowed a negative number to appear by itself as a member of an equation. Cf. O. Neugebauer, *Mathematische Keilschrift Texte*, part 3, page 38 (1937).

The same note terminates with two other incorrect statements in the same sentence. The former of these is to the effect that the real calculating with negative numbers begins with R. Descartes, who died in 1650. As a matter of fact, this statement is directly

contradicted in the corresponding note (149) of the French edition of the same encyclopedia, page 35, where it is asserted that the systematic calculating with negative numbers was posterior to R. Descartes. The second misstatement in the same sentence to which we referred is that R. Descartes assigned to the same letter sometimes a positive and sometimes a negative value. As a matter of fact, a letter had only a positive value in R. Descartes's writings. It should, however, be noted in this connection that by placing a point before a letter R. Descartes indicated that the latter could have both a positive and also a negative value. Cf. J. Tropfke, *Geschichte der Elementar-Mathematik*, volume 2, page 100 (1933).

Although there are fewer actual errors in note 24 of the article cited at the beginning of the present discussion than in note 18 thereof the former involves a statement which is probably more surprising to many readers than any one which appears in the latter, viz., that the positional number system of the Babylonian astronomers to the base 60 involved the use of 59 different numerals. This would correspond to the 9 different digits of our common number system to the base 10, but there seems to be no evidence that it ever existed. It is now well known that the Babylonian number system involves the use of the base 10 up to 60 so that it was unnecessary to have 59 different numerals. In fact the symbols for unity and for 10 are the only ones which were commonly used. Cf. O. Neugebauer, *Vorlesungen über Geschichte der Antiken Mathematischen Wissenschaften*, volume 1, page 4 (1934).

The two opening sentences of note 24 are to the effect that decimal fractions originated during the 16th century and that John Kepler (1571-1630) introduced the decimal comma. It is now known that a decimal comma was used by John Napier in his *Rabdologia* of 1617 and that this is the first known use thereof. The decimal period was used a little earlier by C. Clavius in his *Astrolabium* of 1593. Simon Stevin (1540-1620) was the first to treat decimal fractions systematically and extensively, and he has often been credited with their invention. He devoted a small pamphlet to this subject which appeared in 1585 both separately and as an appendix to his *Practique d'Arithmetique* and he urged the governments to introduce decimal money, weights and measures in order to simplify calculations. In fact, he also pointed out the importance of dividing the angle decimally and thus became the most influential forerunner of the use of decimal systems. Special decimal fractions appeared before the beginning of the 16th century in Arabia.

In note 25, page 22, it is stated that traces of calculating with powers and roots appear already in the 14th to the 16th century. It is now known that such traces appear already in the works of Archimedes and possibly earlier in the work of the ancient Babylonians as was noted by O. Neugebauer on pages 200 and 201 of the history cited at the close of next to the preceding paragraph. It is not implied that all of the historical notes in the *Enzyklopädie der mathematischen Wissenschaften* have become unreliable in the comparatively short time since this work has been before the public. A considerable number of historical statements represent advances in this history of permanent value but there are others which have become antiquated in recent years as a result of the rapid recent progress along historical lines. It may be of interest to emphasize a few definite evidences of this progress by means of a publication which is in general still very favourably known in order to exhibit more clearly the fundamental nature of this progress.

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G. A. MILLER.

On a Property of Self-Sufficient Sets

1. A set (A) of n distinct non-zero positive integers: $a_1, a_2, a_3, \dots, a_n$; where each $a < p$, p being an odd prime, is said to be a self-sufficient p -set, when the product of every two (or more) members of the set is congruent modulo p , with a member of the set itself; i.e. when $a_i a_j \equiv a_m \pmod{p}$, $0 < (i, j, m) \leq n$.

Since any $a^{p-1} \equiv 1 \pmod{p}$, unity must be a member of such a set.

2. The object of this note is to prove the following

THEOREM*: The n members of any self-sufficient p -set (A), are the k -ic residues of p , where $kn = p - 1$.

Let g be a primitive root of p . Then corresponding to the given set (A), there is another set (B) of positive integers: $b_1, b_2, b_3, \dots, b_n$; given by the relation: $g^{b_r} \equiv a_r \pmod{p}$, $1 \leq r \leq n$; and $0 < b_r \leq \phi(p)$.

Evidently the sum of every two (or more) members of (B) is congruent modulo $\phi(p)$ with a member of (B) itself; i.e. $b_i + b_j \equiv b_m \pmod{\phi(p)}$.

Let k denote the greatest common divisor of the members of (B).

* This theorem was suggested to me by Dr. S. Chowla.

Then the numbers: $k, 2k, 3k, \dots, (p-1)$; can be represented modulo $\phi(p)$ as the sums of b 's: and no other non-zero positive number $\leq \phi(p)$ can be so represented. Hence the members of (B) are $k, 2k, 3k, 4k, \dots, (p-1)$; in some order.

Let $b_r = rk$, so that $b_n = nk = (p-1)$.

Then $a_r \equiv g^{rk} \pmod{p}$.

If $g^r \equiv t \pmod{p}$, we get $a_r \equiv t^k \pmod{p}$.

Since there are just n k -ic residues of p and no more, the numbers of set (A) are all the k -ic residues of p .

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The Twisted Straight Line

1. According to a well-known theorem in differential geometry, there exists a curve, unique except for position and orientation in space, whose curvature k and torsion τ are assigned functions

$$\kappa = f(s); \quad \tau = \phi(s)$$

of the arc length s . When $\phi(s) \equiv 0$, we have a plane curve with curvature $f(s)$. I am not sure whether the other case, when $f(s) \equiv 0$ is sufficiently well known among students and teachers. The curve is now a *twisted straight line*, i.e. a straight line with a definite osculating plane and hence also a definite torsion $\phi(s)$ associated with each point. By changing the function $\phi(s)$ we vary the twist of the straight line. A rope which is being twisted under tension is a material symbol of a twisted straight line of nearly constant torsion.

One immediate application of this idea is that every conoidal surface may be considered as the surface of principal normals or surface of binormals of a twisted straight line. Now it is known that the sum of the principal curvatures at a point P of the surface of binormals is given by

$$J = \frac{\kappa^2 + \kappa\tau^2 u^2 - \tau' u}{(1 + \tau^2 u^2)^{3/2}} \quad \dots (1)^*$$

where u is the distance of P from the curve measured along the binormal through it. Putting $k=0$ we have $J = -\tau' u / (1 + \tau^2 u^2)^{3/2}$. This vanishes when the straight line has constant torsion and the binormal surface is in this case a helicoid. Hence the helicoid is a minimal surface.

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A. NARASINGA RAO.

* Weatherburn: *Differential geometry of three dimensions*, Vol. I, (1931) P. 72.

Note on an Inequality

1. The object of this note is to generalise an inequality on a definite integral given in problem 121 in Polya-Szegö: *Aufgaben u. Lehrsätze*, II Abschn. p. 62. The problem there given is equivalent to the following:—

Let $f(a) = f(b) = 0$, $|f'(x)| \leq M$ in $a \leq x \leq b$.

Then
$$\frac{M(b-a)^2}{4} > \left| \int_a^b f(t) dt \right|$$

We propose proving the following generalisation:

If M is the upper bound of $|f'(x)|$ in (a, b) , then

$$|J| \equiv \left| \int_a^b f(t) dt - \frac{1}{2}(b-a) \{f(a) + f(b)\} \right| < \frac{M(b-a)^2}{4} - \frac{1}{4M} \{f(b) - f(a)\}^2 \dots \dots \dots (E)$$

Before proceeding to the proof of the inequality (E), it may be remarked that

$$|J| = \left| \int_a^b \left(\frac{a+b}{2} - x \right) f'(x) dx \right| \leq \frac{M(b-a)^2}{4}$$

which is obviously a much weaker inequality than (E).

2. *Proof of (E):* We have

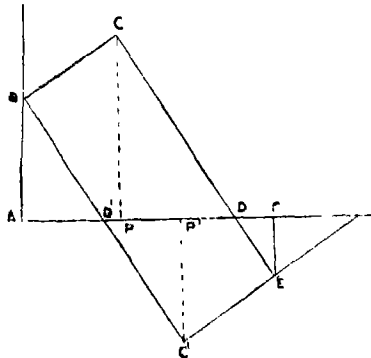
$$\int_a^b f(t) dt - \frac{1}{2}(b-a) \{f(b) + f(a)\} = \int_a^b \left\{ f(t) - \frac{f(b) + f(a)}{2} \right\} dt \\ - \int_a^b \phi(t) dt, \text{ say}$$

where $\phi(t) = f(t) - \frac{1}{2}\{f(b) + f(a)\}$, so that $\phi(a) + \phi(b) = 0$.

Let $\alpha = \phi(a) = -\frac{1}{2}\{f(b) - f(a)\}$. Then the curve $y = \phi(t)$ lies below the lines

$$y = \alpha + M(t-a) \quad (\text{Line BC})$$

$$y = -\alpha - M(t-b), \quad (\text{Line CE})$$



i.e. it lies below the polygonal line BCDE. Similarly it lies above the polygonal line BD'C'E. Hence

$$\int_a^b \phi(t) dt \leq \text{area under BCDE} \\ = \frac{1}{2}(\text{CP} + \text{AB})\text{AP} + \frac{1}{2}(\text{CP} - \text{EF})\text{PF}$$

An easy calculation gives

$$\begin{aligned} \text{CP} &= \frac{1}{2}\text{M}(b-a), \text{EF} = \text{AB} = \alpha, \text{AP} = \frac{1}{2}(b-a) - \frac{\alpha}{\text{M}}, \text{PF} = \frac{1}{2}(b-a) + \frac{\alpha}{\text{M}} \\ \therefore \int_a^b \phi(t) dt &\leq \frac{1}{2} \left[\left\{ \frac{1}{2} \text{M}(b-a) + \alpha \right\} \left(\frac{b-a}{2} - \frac{\alpha}{\text{M}} \right) \right. \\ &\quad \left. + \left\{ \frac{1}{2} (b-a)\text{M} - \alpha \right\} \left(\frac{b-a}{2} + \frac{\alpha}{\text{M}} \right) \right] \\ &= \frac{\text{M}(b-a)^2}{4} - \frac{\alpha^2}{\text{M}} = \frac{\text{M}(b-a)^2}{4} - \frac{1}{4\text{M}} \left\{ f(b) - f(a) \right\}^2 \end{aligned}$$

Similarly, considering the curve BD'C'E we get

$$\int_a^b \phi(t) dt \geq \frac{\alpha^2}{\text{M}} - \frac{\text{M}(b-a)^2}{4}$$

Hence

$$\begin{aligned} \left| \int_a^b \phi(t) dt \right| &= \left| \int_a^b f(t) dt - \frac{1}{2}(b-a)\{f(a) + f(b)\} \right| \\ &\leq \frac{\text{M}(b-a)^2}{4} - \frac{1}{4\text{M}} \left\{ f(b) - f(a) \right\}^2. \end{aligned}$$

It is quite clear that equality can only occur when $\phi(t)$ coincides completely with ECDE or BD'C'E, which is however not possible since $\phi'(t)$ exists at the points P, P'. Hence (E) is proved.

Since we can approximate to BCDE as closely as we like by a curve through B, E, it is obvious that the inequality cannot be improved.

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A Note on the Zeros of $\sum_{r=0}^n \frac{z^r}{r!} = 0$.

In this note it is proposed to shew, by an elementary method, that the zeros of $\sum_{r=0}^n \frac{z^r}{r!}$, for large n , lie in the region $n > |z| > n e^2$.

1. We shall first prove that for all n , the zeros all lie in $|z| < n$.

Let $f_n(z) = \sum_{r=0}^n \frac{z^r}{r!}$, and $\phi(z) = f_n(nz)$

Let $g(z) = z^n \phi\left(\frac{1}{z}\right) = b_0 + b_1 z + \dots + b_n z^n$;

then $b_0 = \frac{n^n}{n!}$, $b_1 = \frac{n^{n-1}}{(n-1)!}$, ... ,

and, obviously, $b_0 = b_1 > \dots > b_n$.

Then all the zeros of $g(z)$ lie outside the unit-circle (Vide e.g. Landau: *Ergebnisse der Funktionentheorie*, p. 26, Satze von Eneström).

As the proof is short and simple we give it here:

$$(1-z)g(z) = b_0 - \{b_0 - b_1\}z + (b_1 - b_2)z^2 + \dots + (b_{n-1} - b_n)z^n + b_n z^{n+1}$$

Hence for $|z| \leq 1$, excluding $z=1$, we have

$$|(1-z)g(z)| \geq b_0 - \{(b_0 - b_1) + (b_1 - b_2) + \dots + (b_{n-1} - b_n) + b_n\} = 0.$$

It is obvious that the right side in the above expression for $(1-z)g(z)$ can be zero only when $z=1$. Hence $|g(z)| > 0$ when $|z| \leq 1$.

Going back now to the function $f_n(z)$, we see immediately that all its roots lie in the region $|z| < n$.

2. At a zero, say z_0 , of $f_n(z)$ we have

$$\frac{z_0^{n+1}}{(n+1)!} + \frac{z_0^{n+2}}{(n+2)!} + \dots = -e^{z_0} \dots \dots \dots (2.1)$$

Also $|z_0| < n$. Let $|z_0| = (n+1)u_0$

Then

$$\begin{aligned} \exp\{-|z_0|\} &< \frac{|z_0|^{n+1}}{(n+1)!} + \dots < \frac{|z_0|^{n+1}}{(n+1)! (1-u_0)} \\ &= \frac{(n+1 \cdot u_0)^{n+1}}{(n+1/e)^{n+1}} \cdot \frac{1+\xi_n}{\sqrt{(2\pi n) \cdot (1-u_0)}} \quad \text{for large } n. \end{aligned}$$

Hence

$$u_0 \cdot e \cdot (1+\xi_n)^{1/(n+1)} \div \{(1-u_0)\sqrt{2\pi n}\}^{1/(n+1)} > e^{-u_0} \dots \dots \dots (2.2)$$

Now by (§ 1) $u_0 < n/(n+1)$, so that $1 < \frac{1}{(1-u_0)} < n+1$
and $1 < (1-u_0)^{-1/(n+1)} < (n+1)^{1/(n+1)} = 1 + \xi'_n$,

Hence from (2.2) we have

$$u_0 e^{u_0+1} > 1 + \xi''_n$$

$$\text{or } 1 + u_0 + \log u_0 > \xi''_n.$$

Now $1 + u_0 + \log u_0$ is a monotonic function which vanishes for one value u_0 in $\frac{1}{e} > u_0 > \frac{1}{e^2}$

Hence when n is large $u_0 > \frac{1}{e^2}$ whence $n > |z_0| > \frac{n}{e^2}$ where z_0 is any root of $\sum_{r=0}^n \frac{z^r}{r!} = 0$.

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Inequalities Connected with a Tetrahedron

In this note are proved certain inequalities connected with a tetrahedron. Most of them are analogous to the inequalities relating to a triangle.

1. Let ABCD be a tetrahedron the areas of whose faces are $\alpha, \beta, \gamma, \delta$ and whose volume is V . Let R, r be the circum- and in-radii, respectively, of ABCD and let S be the total surface of the tetrahedron. If a, a' ; b, b' ; c, c' ; be the lengths of the three pairs of opposite edges of ABCD, we have

$$V = \frac{1}{6} aa'd_1 \sin \theta_1 \leq \frac{1}{6} aa'b$$

where d_1, θ_1 , are respectively the shortest distance and the angle between the pair of opposite edges of lengths a, a' .

Similarly $V < \frac{1}{6} cc' b'$.

From these two we have $V < \frac{1}{6} (aa' bb' cc')^{\frac{1}{2}} \dots \dots (1.1)$

O being the circum-centre of ABCD, if v_1, v_2, v_3, v_4 be the volumes of the four tetrahedra OBCD, OCAD, OABD, OABC,

we have, by (1.1), $v_1 < \frac{R^{3/2}}{6} \sqrt{(a b' c')} < \frac{R^{3/2}}{6} \sqrt{(4 R \alpha)}$.

$$\text{i.e. } v_1 < \frac{R^3}{3} \sqrt{\alpha}.$$

and similar inequalities hold for v_2, v_3 and v_4 . Hence, by adding up, we obtain

$$V < \frac{R^3}{3} (\sqrt{\alpha} + \sqrt{\beta} + \sqrt{\gamma} + \sqrt{\delta}) \dots (1.2)$$

Since, of all triangles, inscribed in a circle, the equilateral triangle has the maximum area and since the circumradius of any tetrahedron is greater than or equal to that of any of its faces, we have

$$R^3 \geq \frac{4A}{3\sqrt{3}} \quad \dots (1.3)$$

where A is the area of the greatest of the faces of the tetrahedron.

2. We next proceed to obtain an inequality connecting the in- and circum-radii of a tetrahedron. It is well known that of all tetrahedra inscribed in a sphere of radius R the one whose volume as also the total surface area is a maximum is the regular tetrahedron and it is easily found that the value of this maximum surface area is $8R^2/\sqrt{3}$. Hence this theorem gives for any tetrahedron, the inequality $R^3 \geq \frac{1}{3} S\sqrt{3}$... (2.1)

The inequality becomes an equality when and only when the tetrahedron is regular. Again from a theorem due to Steinitz which states that "of all tetrahedra of given surface area, the regular tetrahedron has the maximum volume and hence the maximum in-radius" we obtain for any tetrahedron,

$$r^3 \leq \frac{S}{24\sqrt{3}} \quad \dots (2.2)$$

the equality holding when and only when the tetrahedron is regular. From (2.1) and (2.2) we obtain $r/R \leq 1/3$ where, as before the equality holds only when the tetrahedron is regular. The above result is the three-dimensional analogue of the result "the in-radius of a triangle is less than or equal to half of its circum-radius."

... (2.3)

Since the volume V of a tetrahedron is given by the formula $V = rS/3$, we obtain, from (2.2)

$$S^3 - 216 \sqrt{3} V^2 \geq 0. \quad \dots (2.4)$$

This is the three-dimensional analogue of the inequality

$$S^3 - 12\sqrt{3}A \geq 0 \quad \dots (2.5)$$

relating to the perimeter S and the area A of a triangle. It may, perhaps, be useful to observe, here, that the inequalities (2.4) and (2.5) are sharper than those given by Schwarz for a convex solid and a convex oval † viz.

$$S^3 - 36\pi V^2 \geq 0 \text{ and } S^3 - 4\pi A \geq 0.$$

† Blaschke: *Kreis and Kugel* Pp. 30-31 and Pp. 78-79.

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ASTRONOMICAL NOTES

Planet Notes from October to December 1938

(All times mentioned here are I. S. T. and the co-ordinates of planets given are for
5 hours 30 minutes I. S. T.)

Sun. The Sun after passing the equator on 23rd September moves with an increasing South declination and attains its maximum declination South of the equator on the evening of 22nd December thus marking the beginning of winter in the Northern hemisphere. During the period October to December the Sun will be passing from the centre of Virgo to the centre of Sagittarius. Its Right Ascension will be changing from 12 h. 26 m. to 18 h. 38 m. and Declination from $2^{\circ} 50'$ S. to its maximum $23^{\circ} 27'$ S on 22 December and then decreases to $23^{\circ} 10'$ S by the end of December. On 22nd November there will be a partial eclipse of the Sun invisible in India.

The Sun's position at the beginning of each month is as follows :—

	<i>1st October</i>	<i>1st November</i>	<i>1st December</i>
R. A. ...	12 h 26 m	14 h 22 m	16 h 26 m
Dec. ...	$2^{\circ} 51' \text{ S}$	$14^{\circ} 9' \text{ S}$	$21^{\circ} 40' \text{ S}$

Moon. The following phases of the Moon will occur during the period October to December.

	<i>New Moon</i>	<i>1st Quarter</i>	<i>Full Moon</i>	<i>Last Quarter</i>
October 23rd at	2-12 P.M.	{ 1st at 5-15 P.M. 31st at 1-15 P.M.	9th at 3-7 P.M.	16 at 2-54 P.M.
November 22nd at	5-35 A.M.	30th at 9-29 A.M.	8th at 3-53 A.M.*	14 at 9-50 P.M.
December 21st at	11-37 P.M.	30th at 4-23 A.M.	7th at 3-52 P.M.	14 at 6-47 A.M.

The Moon will be nearest the earth on 16th October, 11th November, and 9th December and will be farthest on 2nd, 30th October, 27th November and 24th December. On 8th November there will be a total eclipse of the Moon visible in India. The circumstances of the eclipse will be as follows :—

Beginning of total eclipse	...	3-15 A.M.	I. S. T.
Middle	..	3-56 A.M.	..
End	..	4-38 A.M.	..

Moon's position at the beginning of each month :—

	<i>1st October</i>	<i>1st November</i>	<i>1st December</i>
R. A. ...	18 h 8 m	21 h 5 m	23 h 7 m
Dec. ...	$19^{\circ} 53' \text{ S}$	$11^{\circ} 12' \text{ S}$	$0^{\circ} 44' \text{ S}$

Mercury. The planet with direct motion moves from Virgo to Sagittarius and reaching the stationary point in the constellation on December 24th retraces the path and enters Scorpio on 13th December and remains in it till the end of December. During October Mercury is too near the Sun to be visible but by the end of November it can be seen near the Western horizon for about an hour and a half after sunset. Again by the beginning of December it approaches the Sun and by the end of the month the planet is visible as a morning object rising in the east an hour and a half before the Sun.

The following phenomena of Mercury will take place during this period :—

Mercury stationary	...	4th and 24th December
Superior conjunction with Sun...		10th October
Inferior	...	14th December
Conjunction with Venus	...	9th November
Conjunction with Moon	...	24th November & 20th December
Greatest elongation 22° E	...	25th November.

The position of the planet is as follows :—

		<i>1st October</i>	<i>1st November</i>	<i>1st December</i>
R. A.	...	12 h 2 m	15 h 13 m	17 h 54 m
Dec.	...	1° 46' N	19° 17' S	25° 20' S

Venus. During the beginning of October Venus enters Scorpio and by the end of November it moves to Libra and again moves to Scorpio by the end of December. Venus is an evening object during October setting earlier day by day till it nears the Sun on November when it cannot be seen, but during December it is visible as a morning object. Like Mercury this planet is stationary on two occasions i.e., on 30th October and 9th December and is in inferior conjunction with the Sun on 20th November. Venus attains its greatest brilliancy on 16th October and 26th December. It is in conjunction with the Moon on 26th October and 18th December.

The positions of the planet will be as follows :

		<i>1st October</i>	<i>1st November</i>	<i>1st December</i>
R. A.	...	15 h 11 m	16 h 7 m	15 h 16 m
Dec.	...	22° 39' S	26° 54' S	18° 23' S

Mars. This planet moves with a direct motion from Leo to Libra during this period. Mars is easily visible as a morning object on the eastern sky during the early morning hours. Mars is in conjunction with the Moon on 21st October, 18th November and 17th December. The planet is also in conjunction with the planet Neptune on 12th October.

The positions of the planet will be as follows:—

	<i>1st October</i>	<i>1st November</i>	<i>1st December</i>
R. A. ...	11 h 5 m	12 h 17 m	13 h 27 m
Dec. ...	7° 8' N	0° 38' S	7° 59' S

Jupiter. The planet with a retrograde motion enters Capricornus from Aquarius on 2nd October and remaining in this constellation for about a month retraces its path with direct motion and again moves to Aquarius on the 4th November. Jupiter is visible throughout this period as an evening object. Jupiter is in conjunction with the Moon on the following dates 5th Oct., 2nd Nov., 29th Nov. and 27th Dec. and is stationary on the 19th October.

The positions of the planet will be as follows:—

	<i>1st Oct.</i>	<i>1st Nov.</i>	<i>1st Dec.</i>
R. A. ...	21 h 43 m	21 h 42 m	21 h 52 m
Dec. ...	15° 1' S	15° 2' S	14° 5' S

Saturn. This planet is visible throughout the night during October and during the latter part of December it will be visible till midnight since it sets by midnight at the end of December. Saturn will be in Pisces throughout. Saturn is in opposition with the Sun on 8th October and in conjunction with the Moon on 9th October, 5th November, 3rd and 30th December. The planet is stationary on 15th December. The positions of the planet will be as follows:—

	<i>1st Oct.</i>	<i>1st Nov.</i>	<i>1st Dec.</i>
R. A. ...	1 h 0 m	0 h 52 m	0 h 46 m
Dec. ...	3° 27' N	2° 34' N	2° 5' N

Uranus. This planet will be visible shortly after 8 P.M. during October and by the end of December it will be visible only till 2 hours after midnight. On 8th Nov. Uranus will be in opposition with the Sun.

The positions of Uranus will be as follows:—

	<i>1st October</i>	<i>1st November</i>	<i>1st December</i>
R. A. ...	3 h 0 m	2 h 55 m	2 h 50 m
Dec. ...	16° 37' N	16° 18' N	15° 57' N

Neptune. This planet will be visible only during the morning hours. It is stationary on the 26th December.

	<i>1st October</i>	<i>1st November</i>	<i>1st December</i>
R. A. ...	11 h 31 m	11 h 34 m	11 h 37 m
Dec. ...	4° 23' N	4° 0' N	3° 46' N

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SOLUTIONS TO QUESTIONS

Question 1607

(R. RAMANUJAM):—Prove the theorem of Pythagoras without constructing squares on the sides of the triangle.

Solution by A. Ranganatha Rao.

Let ABC be a triangle, right-angled at A. Draw AF perpendicular to BC and produce it to D such that AD=BC. Draw DE perpendicular to AC.

The two right-angled triangles BAC, AED are congruent.

∴ CA = DE and AB = AE.

$$\begin{aligned}\text{Quad. } ABDC &= \triangle ABD + \triangle ADC. \\ &= \frac{1}{2} AC \cdot DE + \frac{1}{2} AB \cdot DE \\ &= \frac{1}{2} (AC^2 + AB^2).\end{aligned}$$

$$\text{Also quad. } ABDC = \frac{1}{2} BC \cdot AD = \frac{1}{2} BC^2.$$

$$\text{Hence } BC^2 = AB^2 + AC^2.$$

The well-known Bhaskara-Wallis' solution by the use of similar triangles FBA and FAC is given by

T. Manivanan, T. K. Ragavachari and the proposer.

Question 1745

(D. R. KAPREKAR):—We have $91 \times 819 = 74529$ and also $9901 \times 980199 = 9704950299$ where the corresponding numbers in the second multiplication are obtained by inserting 9 and 0 alternately before the first digit and between the successive digits of all the numbers in the first multiplication. It will be found that the multiplication is still true if we insert alternately blocks of 99 and 00 or 999 and 000 and so on.

It is required to find the principle underlying the result and to find other number pairs like 91 and 819 with the same property.

Solution by V. Narasimhamurti.

Let $10 - a_1, a_1, 10 - a_2, a_2, \dots$ and $10 - b_1, b_1, 10 - b_2, b_2, \dots$ be the digits in order from the left of the two given numbers of m and n digits. Hence the given numbers are

$$\{(10 - a_1) 10^{m-1} + a_1 10^{m-2} + \dots\} \text{ and } \{(10 - b_1) 10^{n-1} + b_1 10^{n-2} + \dots\}$$

Let their product be the number of l digits

$$(10 - c_1) 10^{l-1} + c_2 10^{l-2} + (10 - c_3) 10^{l-3} + \dots$$

where $10 - c_1, c_2, \dots$ are the digits from the left in the product. A general statement of the problem would require that the following equality will hold for every integer $s > 0$, viz.,

$$\{(10^s - b_1)(10^s)^{n-1} + b_2(10^s)^{n-2} + \dots\} \{(10^s - a_1)(10^s)^{m-1} + a_2(10^s)^{m-2} + \dots\} \\ = (10^s - c_1)(10^s)^{l-1} + c_2(10^s)^{l-2} + \dots \quad \dots \text{ I}$$

Putting $10^s = x$, I can be written in the form

$$\{x^m - a_1 x^{m-1} + (a_2 + 1)x^{m-2} - + \dots\} \{x^n - b_1 x^{n-1} + (b_2 + 1)x^{n-2} - + \dots\} \\ = \{x^l - c_1 x^{l-1} + (c_2 + 1)x^{l-2} - + \dots\} \quad \dots \text{ II}$$

So that we have $m + n = l$

$$c_1 = a_1 + b_1$$

$$c_2 + 1 = a_2 + 1 + b_2 + 1 + a_1 b_1$$

$$c_3 = a_3 + b_3 + a_1(b_2 + 1) + b_1(a_2 + 1)$$

$$c_4 + 1 = a_4 + 1 + b_4 + 1 + a_1 b_3 + a_3 b_1 + (a_2 + 1)(b_2 + 1)$$

etc.

Since the digit in the first place must be greater than zero,

$$10 > a_1, b_1, c_1 > 0.$$

The digits in the other places are such that

$$10 > a_{2r}, b_{2r}, c_{2r} \geq 0; 10 \geq a_{2r+1}, b_{2r+1}, c_{2r+1} > 0$$

we can now construct any number of examples with the required property.

Let us take 71 as the multiplicand, and let $(10 - b_1) 10^2 + b_2 10 + (10 - b_3)$ be its multiplier. The product is

$$(10^2 - 3 \cdot 10 + 1) \{10^3 - b_1 10^2 + (b_2 + 1) 10 - b_3\}.$$

Putting $x = 10$ we get as in II

$$(x^2 - 3x + 1) \{x^3 - b_1 x^2 + (b_2 + 1)x - b_3\} \\ \equiv x^5 - c_1 x^4 + (c_2 + 1)x^3 - c_3 x^2 + (c_4 + 1)x - c_5,$$

where $10 - c_1, c_2, 10 - c_3, c_4, 10 - c_5$ are the digits of the product from the left.

Our limitations then are

$$10 > c_1 = 3 + b_1 \geq 1$$

$$10 > c_2 = -1 + b_2 + 1 + 3b_1 + 1 \geq 0$$

$$10 \geq c_3 = b_3 + 3b_2 + 3 + b_1 \geq 1.$$

$$10 > c_4 = -1 + 3b_3 + b_2 + 1 \geq 0.$$

$$10 \geq c_5 = b_3 \geq 1.$$

Subject to these inequalities we may take

$$(1) \quad b_1 = 2, \quad b_2 = 1, \quad b_3 = 2;$$

whence the product becomes $71 \times 818 = 58078$;

$$(2) \quad b_1 = 2, \quad b_2 = 1, \quad b_3 = 1;$$

whence the product becomes $71 \times 819 = 58149$;

$$(3) \quad b_1 = 2, \quad b_2 = 0, \quad b_3 = 1$$

whence we have $71 \times 809 = 57439$.

Other pairs of numbers can be obtained in the same way

Eg. $82 \times 708 = 58056$; $918 \times 709 = 650862$.

Remarks by the Proposer.

The Solution given by J. Mahrenholz in Vol. V, No. 4, p. 188 is only an algebraic rendering of the question, while in the other examples given by him the figures to be inserted are not 9 and 0 as required in the question.

Though I have not succeeded in getting a general formula, I have by continuous thinking over the matter for over a dozen years obtained empirically nearly a hundred such pairs with the property stated in the question of which I give a few below:—

$$81 \times 729 = 59049$$

$$71 \times 819 = 58149$$

$$81 \times 819 = 66339$$

$$92 \times 729 = 67068$$

In the two pairs given below, we have to regard groups of two digits as units between which groups of 9's and 0's are to be inserted. Thus

$$94 \ 03 \ 92 \times 93 \ 02 = 87 \ 47 \ 52 \ 63 \ 84$$

$$\underline{994003992} \times \underline{993002} = \underline{987047952063984}$$

We may also insert r nines and r zeros in the gaps instead of single nines and zeros.

$$\text{Again } 97 \ 01 \ 93 \times 98 \ 01 \ 92 = 95 \ 09 \ 75 \ 41 \ 70 \ 56$$

is another product with the same property.

Question 1756

(D. R. KAPREKAR)—Prove without actual multiplication that the last nine digits of huge number N given below are 999977778.

$$N = (546194538)\{1 + (24580) + (24580)^2 + \dots + (24580)^{10}\},$$

Solution by R. Sharangi.

$$\begin{aligned} N &= 546194538 \times \{1 + 24580 + (24580)^2 + \dots + (24580)^{10}\} \\ &= 546194538 \times \left\{ \frac{(24580)^{11} - 1}{(24580) - 1} \right\} \quad \bullet \\ &= \frac{546194538}{24579} \times \left\{ (24580)^{11} - 1 \right\} = 22222 \left\{ (24580)^{11} - 1 \right\} \\ &= \text{a number ending in 11 zeros} - 22222. \end{aligned}$$

Hence the last nine digits of N are
999, 977, 778.

Also solved by H. Z. Gilami, B. Venkataramana Kolla, V. Narasimhamurti and J. Mahrenholz.

Question 1759 (Corrected)

(K. RANGASWAMI):—If $x > 1$ and r, s are positive integers, then

$$\frac{x^r - x^{-r}}{r} > \frac{x^s - x^{-s}}{s}$$

according as $r > s$.

Solution by B. R. Venkataraman.

Let r be a positive integer. Then if $x > 0$ and $0 < k \leq 2r - 1$, we have

$$(x^{2r-k} - 1)(x^{k+1} - 1) > 0.$$

$$\text{i.e., } x^{2r+1} + 1 > x^{2r-k} + x^{k+1}$$

Giving the values $0, 1, 2, \dots, r-1$, for k in succession and adding up we get

$$r(x^{2r+1} + 1) > x + x^2 + \dots + x^{2r}.$$

Hence, if $x > 1$, so that $x - 1$ is positive we have

$$r(x - 1)(x^{2r+1} + 1) > x^{2r+1} - x.$$

$$\text{i.e. } (r + 1)(x^{2r+1} - x) < r(x^{2r+2} - 1)$$

$$\text{i.e. } \frac{x^{2r+1} - x}{r} < \frac{x^{2r+2} - 1}{r + 1}.$$

Multiplying both sides of the inequality by $x^{-(r+1)}$ we get,

$$\frac{x^r - x^{-r}}{r} < \frac{x^{r+1} - x^{-(r+1)}}{r + 1}.$$

It follows immediately from the above that, when $x > 1$

$$\frac{x^r - x^{-r}}{r} > \frac{x^s - x^{-s}}{s} \quad \text{according as } r > s.$$

If we substitute $\frac{1}{x}$ for x in the above result the inequalities are reversed.

Thus, if $x < 1$, $\frac{x^r - x^{-r}}{r} < \frac{x^s - x^{-s}}{s}$ according as $r > s$.

Also solved by C. Radhakrishna Rao and R. Sharangi.

Question 1762

(R. RAMANUJAM).—If p , q , r represent the lengths of the altitudes of a triangle, find an expression for the area in terms of p , q , and r .

Solution by Johannes Mahrenholz.

Let a , b , c be the sides of the triangle and Δ its area.

Then $ap = bq = cr = 2\Delta$

Now, $16\Delta^2 = (a+b+c)(-a+b+c)(a-b+c)(a+b-c)$

$$= 16\Delta^2 \left(\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \right) \left(-\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \right) \left(\frac{1}{p} - \frac{1}{q} + \frac{1}{r} \right) \left(\frac{1}{p} + \frac{1}{q} - \frac{1}{r} \right).$$

$$\therefore \Delta = p^2 q^2 r^2 \div \{(pq + qr + rp)(-pq + qr + rp)(pq - qr + rp)(pq + qr - rp)\}^{\frac{1}{2}}$$

Also solved by B. R. Venkataraman, C. Radhakrishna Rao, R. Sharangi, K. S. Ranganathan, S. N. Kawalgikar and analytically by B. G. Pandse.

REVIEWS

J. TINBERGEN: *An Econometric Approach to Business Cycle Problems*: Hermann & Co., Paris, 1937, 73 pp. 18 francs.

The usefulness of the mathematical method in economic research has not yet been fully realised in India, but in the West almost all the work in the field is done with this powerful tool. The book under review is an attempt at a mathematical treatment of the business cycle problems of the Netherlands.

The first of the three chapters of the book introduces the variables involved in the analysis. There are thirty-two of these consisting of prices, quantities, and values. They are connected among themselves by twenty-two regression equations which have been fitted in by the partial time regression method. These equations are valid only for the short run for, in their construction, as the author admits, long run influences such as the effect of new investments on productive capacity have been ignored. Again, it is assumed that workers spend the whole of their income whereas non-workers spend only a part of it. The most important of the conclusions reached by the author seems to be that the interest rate has not exerted much influence on the volume of investment. Investments and savings are found to be independent of each other for the short period but the author agrees with Mr. Keynes with regard to the long run relationship between the two.

The second chapter draws attention to the various problems in the history and the theory of business cycles that could be studied with the help of the variables and equations of the first chapter, and gives a detailed discussion of the relative importance of national and international influences in the theory. The author's conclusion is that the chief cyclical movements noticed are the imported ones.

The last chapter is concerned with business cycle policy and it is here that the author is at his best. He examines the various possible ways in which a Government might combat a depression. If the international economic position undergoes no change, the alternatives suggested are Investments on public works, Rationalisation, Lowering of monopoly prices, Wage reductions or Devaluation and the conclusion reached is that devaluation is the most favourable policy at least for the short run. If on the other hand international changes have to be reckoned with, the choice lies between stable exchange rates and stable prices. The latter will involve manipulation of public investment so as to make the total volume non-cyclical. The former leads to manipulation of exchange rates so as to equalise the price index for imports with its own trend value. The stabilising effect of the manipulation of exchange is found to be larger than that of the other. Coming to the question of the best policy, that is, the stabilisation of the level of employment or more generally the attainment of "the maximum of collective opheimity" the author suggests changes in the existing coefficients wherever possible and the introduction of new coefficients like cost of living clauses in wage contracts. However he leaves the question open and concludes that his study only suggests a programme for further work and that what he has offered here must be regarded only as provisional findings.

The attention of the reader may now be drawn to one or two errors of omission and commission found in the book. The regression coefficients given have not been tested for significance and some of them are too small to permit any valid conclusions to be drawn from them. Secondly, many of the equations appear to have been over-simplified at the cost of accuracy, for instance, on pages 22 and 30. Thirdly in Chapter II on page 58, the author's conclusion that the movements of income consist of two cyclical components is not correct. The component dependent wholly on internal economy is given by $c_1x_1' + c_2x_2'$ where x_1 and x_2 are the roots of the equation $x^2 - 15x - 26 = 0$. It is clear that both of them are real but since they are proper fractions, this component becomes insignificant with the passage of time and the other one predominates. The fact is that the cyclical movement is entirely imported from abroad and this may entail changes in the arguments of the first part of Chapter III in the book.

Notwithstanding these defects, the reviewer has no doubt that this admirable treatise will be of great benefit to workers in the field of Trade Cycle theory.

Annamalainagar.

S. SUBRAMANIAN.

M. V. JAMBUNATHAN. M.A., *Bala Thrikonamitti*, M. V. SESHADRI & Co., Bangalore, X + 100 pp. Price Annas Twelve.

This excellent booklet on Elementary Trigonometry by Rashtra Bhasha Visarada M.V. Jambunathan, Lecturer in Mathematics in the Intermediate College, Bangalore, seeks to give an introductory course in the subject up to the solution of right-angled triangles. The exposition is clear and illustrated with diagrams and worked examples, while there are a fair number of graded problems for the learner. The book should prove exceedingly useful to students preparing for the Mysore S.S.L.C. for whom it is primarily intended.

The preparation of technical books in the several Indian vernaculars is the first step towards the percolation of present-day scientific knowledge and experience to the masses and a powerful impetus to this movement is the progressive vernacularisation of the High School and the pre-Graduate courses which are being attempted in the various parts of the country today. While this makes for a more intimate acquaintance with the subject-matter, one should not altogether lose sight of the transition to a different language which is inevitable for one pursuing more advanced courses. A careful choice of notation and terminology with an eye on the whole of the country can help in softening this transition, and it is gratifying to note that the author has recognised this need. Occasionally, however, one meets a phrase which suggests something quite different, e.g. "Samakona Trikona" which suggests an equilateral instead of a right-angled triangle.

For an elementary course, such as is intended by the author, the introduction of the general angle and its trigonometric ratios may well have been omitted. Possibly its inclusion is the result of the author re-printing the first few chapters of a bigger book under contemplation.

A table of natural signs, cosines and tangents at the end, a glossary of technical terms at the beginning enhance the value of the book.

A. NARASINGA RAO.

BOOKS RECEIVED FOR REVIEW

- George Abraham MILLER: *Collected Works, University of Illinois, Urbana, Illinois.*
- MILNE-THOMSON: *Theoretical Hydrodynamics*, Macmillan & Co., London.
Actualites Scientifiques et Industrielles, Hermann et Cie, Paris.
- No. 552. Georges HOSTELET: *Les fondements experimentaux de l'Analyse mathematique des faits statistiques.*
- No. 565. Moshen HACHTROUDI: *Les Espaces d'elements a connexion projective Normale.*
- No. 570. George VALIRON: *Sur les Valeurs exceptionnelles des fonctions meromorphes et de leurs derivees.*
- No. 571. Florin VASILESCO: *La notion de capacite.*
- No. 578. C. de La Vallee POUSSIN: *Les nouvelles methodes de la Theorie du Potentiel et le Probleme generalise de Dirichlet.*
- No. 585. Georges HOSTELET: *Le concours de l'Analyse mathematique à l'Analyse experimentale des faits statistiques.*
- No. 589. Albert LAUTMAN: *Essai sur l' unite des sciences mathematiques dans leur developpement actuel.*
- No. 590. Albert LAUTMAN: *Les schemas de structure.*
- No. 591. Albert LAUTMAN: *Les schemas de genèse.*
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ANNOUNCEMENTS AND NEWS

The following gentlemen have been elected members of the Society:—

J. L. Sharma, M.A., Lecturer, Christ Church College, Cawnpore.

Jageshwar Dayal Vaish, M.A., B.Sc., Teacher, Sadul High School, Bikaner.

D. N. Moghe, Research Scholar, Fergusson College, Poona.

The Managing Committee of the Indian Mathematical Society has been reconstituted as follows for 1939-40.

President : *Dr. R. P. Paranjpye, M.A., D.Sc.*

Treasurer : *Prof. L. N. Subramaniam, M.A.*

Librarian : *Prof. R. P. Shintre, M.A.*

Other Members

1. *Dr. R. Vaidyanathaswamy, M.A., D. Sc., F.R.S.E.*
2. *The Hon'ble Dr. Sir Shah Mohamad Sulaiman, Kt., M.A., LL.D., D. Sc.*
3. *Dr. A. N. Singh, M. Sc., D. Sc.*
4. *Prof. Arunachala Sastri, M.A.*
5. *Prof. A. Narasinga Rao, M.A., L.T.*
6. *Dr. F. W. Levi.*
7. *Dr. P. L. Srivastava, M.A., D. Phil.*
8. *Dr. S. Chowla, M.A., Ph. D.*
9. *Dr. Ram Behari, M.A., Ph. D.*

With the financial co operation of the Punjab University, the Indian Mathematical Society is printing Dr. Hansraj Gupta's Table of Partitions giving the values of $p(n)$ up to $n=600$ and also the values of (n, m) , the number of partitions of n of which the smallest element that occurs is m , for values of n up to 300. This table has enabled Dr. Chowla to show that Ramanujan's conjecture regarding the divisibility of $p(n)$ by q^a where $q=5, 7$ or 11 and $24n-1$ is divisible by q^a , is not true when $q=7, a=3$ (Vide: J. L. M. S. Vol. 9, p. 247).

The University of Illinois Press recently published Volume II of the Collected Works of Professor G. A. Miller. This volume includes an article of 36 pages on primary facts of the history of Mathematics and the correction of errors in text-books on this history, but the greater part of the volume is devoted to the publications of its author during the years from 1900 to 1908. These are practically confined to the theory of groups of finite order. This is the first time that an American university published the collected works of one of its faculty relating to pure mathematics.

We regret very much to announce the death on 12-11-1938 of Pandit Hemraj, M.A., Principal of the Dayal Singh College, Lahore. Pandit Hemraj has been a member of the Indian Mathematical Society since 1915 and was an active member of its managing committee for over 13 years.

QUESTIONS FOR SOLUTION

1765. (D. R. KAPREKAR). If A, B, C, D are four integers each containing the same numbers of digits such that $A+C=(9)^l$ and $B+D=(9)^l$, then

$$(A \times B) + (C \times D) + (A \times D) + (B \times C) + A + B + C + D = (9)^{2l}.$$

[The symbol $(9)^l$ is the integer formed by the digit 9 repeated l times.]

1766. (S. S. PILLAI). There are r bags each containing a fixed number of balls. By taking at most t bags at a time, we want to form groups containing any number of balls from 1 to n . Let $f(t, n)$ denote the least value of r for which this is possible.

Show that $f(t, n) = [(n-1)(2^t-1)] + t$.

1767. (W. F. KIBBLE): Evaluate

$$\begin{vmatrix} \frac{1}{k} & \frac{1}{k+a_1} & \frac{1}{k+a_1+a_2} & \cdots & \frac{1}{k+\sum_{i=1}^n a_i} \\ \frac{1}{k+b_1} & \frac{1}{k+a_1+b_1} & \frac{1}{k+a_1+a_2+b_1} & \cdots & \frac{1}{k+b_1+\sum_{i=1}^n a_i} \\ \frac{1}{k+b_1+b_2} & \frac{1}{k+a_1+b_1+b_2} & \frac{1}{k+a_1+a_2+b_1+b_2} & \cdots & \frac{1}{k+b_1+b_2+\sum_{i=1}^n a_i} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{k+\sum_{i=1}^n b_i} & \frac{1}{k+a_1+\sum_{i=1}^n b_i} & \cdots & \cdots & \frac{1}{k+\sum_{i=1}^n a_i+\sum_{i=1}^n b_i} \end{vmatrix}$$

and use the result to find

$$\begin{vmatrix} \frac{1}{k} & \frac{1}{k+1} & \frac{1}{k+2} & \cdots & \frac{1}{k+n} \\ \frac{1}{k+1} & \frac{1}{k+2} & \frac{1}{k+3} & \cdots & \frac{1}{k+n+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{k+n} & \frac{1}{k+n+1} & \cdots & \cdots & \frac{1}{k+2n} \end{vmatrix}$$

and each of its minors.

1768. (R. RAMANUJAM). Construct geometrically a triangle given the lengths of the internal bisectors of the three angles.

1769. (A. NARASINGA RAO). ABC , $A'B'C'$ are two variable triangles all of whose sides tend to zero. It is given that the limits $\frac{AB}{A'B'}$, $\frac{BC}{B'C'}$, $\frac{CA}{C'A'}$, exist and are all equal to l . Show by examples, that the conclusion (suggested by similar triangles) that the limit of (area ABC /area $A'B'C'$) exists and is equal to l^2 is not valid. Obtain the further conditions to be fulfilled in order that the result may be true.

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A GENERAL EXTENSION OF FEUERBACH'S THEOREM

BY

K. RANGASWAMI, *Annamalainagar*

1. The interesting theorem in the geometry of the triangle named after Feuerbach* viz., that the nine points circle of a triangle touches its in- and ex-circles, has been shown to be a particular case of various other results. But many of the plane extensions of Feuerbach's result could be derived as particular cases of a more general theorem which, besides, has other important applications to the theory of the Orthopolar Circles of the triangle. The Theorem in question is:

THEOREM I:—If Γ be a range of conics and Γ' any other range formed by conics confocal with those of Γ , the mutual orthopolar circles of corresponding confocals in Γ and Γ' belong to a Coaxal system.

2. It is well known that the totality of conics belonging to the net determined by three linearly independent conics can be put into (1-1) correspondence with the points of a projective plane π . Taking for the three linearly independent conics, the absolute conic Ω (consisting of the pair of circular points of the plane) and any two conics Σ_1, Σ_2 of the range Γ , it is evident that in this representation, the conic Ω will be represented by a point Ω and the conics of the range Γ by the points of the line, say Γ , joining the representative points Σ_1, Σ_2 in π . Now, a confocal system of conics is a range in which the absolute conic Ω is a member. Since the line joining any point Σ' in π to Ω will meet Γ in a point Σ , it follows that the points of the plane π correspond to the totality of conics confocal with those of Γ . Thus, any other line Γ' in π will represent a range of conics confocal with the conics of the range Γ . Further, a pair of corresponding confocals Σ in Γ and Σ' in Γ' will be represented in the π plane by points Σ, Σ' collinear with Ω and the point of intersection, say σ , of the lines Γ and Γ' will represent the self corresponding member σ in the two ranges.

* Vide Casey *Analytical Geometry*. pp. 125-6.

† A range of conics is a system touching four lines.

Let Σ'' be the harmonic conjugate of Ω w. r. t. the corresponding points Σ, Σ' . It is obvious that the locus of Σ'' is a line Γ'' through σ . Thus, the conics Σ'' , which are the harmonic conjugates of the absolute Ω w. r. t. a pair of associated confocals like Σ, Σ' belong to a range Γ'' to which also belongs the self corresponding member σ .

Now, the F-conic of two conics α, β passes through the points of intersection of the pairs of conics like $\alpha + \lambda\beta = 0$ and $\alpha - \lambda\beta = 0$, in the range determined by them. Thus the F-conic of Ω and Σ'' in the confocal system (Σ, Σ') must pass through the common points of Σ and Σ' . In other words, the circle passing through the intersections of the confocals Σ and Σ' , which is their mutual orthoptic circle is the director circle of Σ'' . But, since the conics Σ'' belong to a range, their director circles are coaxal. It follows, therefore, that the mutual orthoptic circles of pairs of corresponding confocals belong to a coaxal system.

3. We next proceed to derive the well-known extensions of Feuerbach's theorem as special cases of Theorem I.

Let ABC be a triangle and Ω the absolute conic consisting of the pair of circular points. Now, the conics in the plane having ABC for a self polar triangle and out-polar to Ω belong to a pencil δ and are the rectangular hyperbolas through the in- and ex-centres of the triangle. Hence, any conic Σ inscribed in the triangle is in-polar to all the conics of the pencil δ ; so also is any conic Σ' confocal with Σ .

If now, Σ' is a point pair (P, P') P and P' are a pair of foci of Σ and are conjugate points w. r. t. the conics of the pencil δ . In particular, P, P' are conjugate w. r. t. the three line pairs of δ viz., the pairs of bisectors of the angles A, B, C. Thus, AP, AP'; BP, BP'; CP, CP' are equally inclined to the bisectors of the angles A, B, C. Therefore P, P' are isogonal conjugates w. r. t. the triangle. We have thus:

LEMMA I:—*A pair of isogonally conjugate points w. r. t. a triangle are conjugate w. r. t. the pencil of rectangular hyperbolas through the in- and ex-centres of the triangle.*

Let Σ be any conic inscribed in the triangle and let its director circle cut the circumcircle of the triangle in X and X'. Since ABC is one triangle inscribed in the circumcircle and circumscribed to Σ , there will be an infinity of such triangles. Thus, if XY, XZ are

tangents to Σ through X meeting the circumcircle again in Y, Z then YZ touches Σ . Further, since X is a point on the director circle of Σ , XY and XZ are at right angles; therefore YZ passes through the circumcentre of ABC . Hence, corresponding to each point X or X' we have a tangent to Σ through O . If, however, Σ passes through O , the two tangents to Σ through O coincide; Consequently the points X and X' will also coincide so that the director circle of Σ touches the circumcircle of the triangle.

Now, the polar circle of the triangle is out-polar to any conic inscribed in the triangle and, therefore, cuts orthogonally its director circle. Hence the director circle of Σ is inverted into itself w. r. t. the polar circle. But the inverse of the circumcircle w. r. t. the polar circle is the nine points circle. Hence, the director circle of Σ touches also the nine points circle we have thus:

LEMMA II:—*If a conic inscribed in a triangle passes through the circumcentre, its director circle touches the nine points circle of the triangle.*

Consider a range Γ of conics inscribed in a triangle and let Γ' be any other range formed by conics confocal with those of Γ . Now, since the conics of the pencil δ are out-polar to those of the range Γ and to the absolute Ω , they are also out-polar to the conics of the range Γ' . In particular, the singular members of Γ' viz., the pairs of opposite vertices, say, X, X' ; Y, Y' ; Z, Z' of the quadrilateral formed by their common tangents will be inpolar to the pencil δ . Thus X, X' ; Y, Y' ; Z, Z' are by Lemma I, isogonal conjugates w. r. t. the triangle and are also the foci of the corresponding confocals in Γ . If we call a quadrilateral whose three pairs of opposite vertices are isogonal conjugates w. r. t. a triangle, an *isogonal quadrilateral* of the triangle, we infer that:

THEOREM II:—*With every range Γ' of conics inscribed in an isogonal quadrilateral of a triangle, it is possible to associate a unique range Γ of conics inscribed in the triangle such that to every conic Σ in Γ there corresponds a confocal Σ' in Γ' .*

Now, the mutual orthoptic circle of (X, X') and the corresponding inscribed confocal, is the pedal circle of X, X' w. r. t. the triangle. Hence we have by Theorem I and lemma I:

THEOREM III:—*If two pairs of opposite vertices of a quadrilateral be isogonal conjugates w. r. t. a triangle, then so will be the third pair and the pedal circles of the three pairs w. r. t. the triangle are coaxial.*

4. Let X, X' be a pair of isogonal conjugates w. r. t. a triangle, O any point on XX' and O' its isogonal conjugate. Now, since the point pairs X, X' ; O, O' are both inpolar to δ all the conics of the range Γ' determined by them viz., the conics touching XX' at O and touching $O'X, O'X'$ will also be inpolar to δ . Hence, by theorem II there is a range Γ of conics confocal with those of Γ' and inscribed in the triangle. Further there is a unique conic σ belonging to both the ranges which touches XX' at O . Now, since σ is a self corresponding member in the two ranges, the mutual orthoptic circle of σ with itself is its director circle. Therefore, by theorems I and III the director circle of σ and the pedal circles of O and X are coaxal. We have, thus, Mr. S. Narayanan's result*:

If an axis of a conic inscribed in a triangle passes through a given point O , the auxiliary circle of the conic corresponding to this axis, the pedal circle of O w. r. t. the triangle and the director circle of the inconic touching the axis at O are coaxal.

In the special case when O is the circumcentre of the triangle, the pedal circle of O is the nine points circle. By Lemma II, the director circle of the conic σ in the previous case touches the nine points circle. Hence, the pedal circle of P, P' also touches the nine points circle of the triangle. This is the well-known extension of Feuerbach's theorem** by M' Cay, viz.,

If an axis of an inconic of a triangle passes through the circumcentre, the corresponding auxiliary circle of the conic touches the nine points circle.

Let Σ' be a conic of Γ' touching PP' at O and Σ the corresponding inscribed confocal in Γ . By Theorems I and III, the mutual orthoptic circle of Σ, Σ' , the director circle of σ and the pedal circle of O are coaxal. If, however, O be the circumcentre we know by Lemma II that the director circle of σ touches the nine points circle. Therefore, the mutual orthoptic circle of Σ, Σ' touches the nine points circle. Thus we have proved Mr. Ramaswamy Aiyar's Theorem† viz.,

If a conic be inscribed in a triangle and a confocal to it pass through the circumcentre, the mutual orthoptic circle of the two conics touches the nine points circle.

* Vide *An Extension of Feuerbach's Theorem* J. I. M. S. Vol. X, p. 419.

** Casey *loc cit* p. 329.

† *Proc. Edin. Math. Soc.* Vol. XV, pp. 74-75.

In the particular case when the confocal consists of a pair of foci of the inconic we get M'Cay's Theorem.

5. We now pass on to apply theorems I and III to the theory of orthopolar circles of the triangle.

Let l be any line in the plane of a triangle ABC ; D, E, F the mid-points of the sides and A', B', C' the projections on l from A, B, C . Now, there is a circle with centre D passing through $B' C'$; another with centre E passing through C, A' and a third with centre F passing through A', B' . These three circles in general have a unique circle L which cuts them all orthogonally. This circle has been called the orthopolar circle of the line l w. r. t. the triangle ABC . It can be shown* that:

LEMMA III:—If Σ be a conic inscribed in a triangle and Σ' the unique confocal to Σ touching l the mutual orthoptic circle of Σ, Σ' cuts orthogonally the orthopolar circle of l w.r.t. the triangle.

As a particular case, when Σ' consists of a pair of foci of Σ we have

LEMOYNE'S THEOREM:—The pedal circle of any point on the line l w.r.t. the triangle cuts orthogonally the orthopolar circle of l .

Let $X, X'; Y, Y'; Z, Z'$ be the three pairs of opposite vertices of an isogonal quadrilateral (of a triangle) whose sides are p, q, r, s and P, Q, R, S their orthopolar circles. Further, let X, Y, Z lie on p, Y, Z', X' on q, X, Z', Y' on r and X', Y', Z on s . By lemma IV, the pedal circles of X, Y, Z cut orthogonally the orthopolar circle P , the pedal circles of Y, Z', X' cut orthogonally the circle Q , etc. But, since a pair of isogonal conjugates determine the same pedal circle, the circles P, Q, R, S all cut orthogonally the pedal circles of X, Y, Z . By theorem III these pedal circles are co-axial. It follows, therefore, that the circles P, Q, R, S are also coaxial. We have thus:

THEOREM IV:—The orthopolar circles of the sides of an isogonal quadrilateral w. r. t. the triangle belong to a coaxial system.

Consider the range Γ' of conics inscribed in an isogonal quadrilateral whose sides are p, q, r, s . By theorem II there is an associated range Γ of confocal conics inscribed in the triangle,

* Vide The Orthopolar Theory. By V. Ramaswamy Aiyar. *J. I. M. S.* Vol. IV pp. 213-215.

which will have besides its sides another line m , for a common tangent. Let m meet the sides of the triangle in A_1, B_1, C_1 .

Now, the pair of points A, A_1 together constitute a conic Σ inscribed in the triangle; there is, therefore, a corresponding confocal conic Σ' in I' . Hence A, A_1 will be a pair of foci of Σ' . By theorems I and III their mutual orthoptic circle i.e., the auxiliary circle of Σ' corresponding to this pair of foci will belong to the coaxal system determined by the pedal circles of X, Y, Z . But, the auxiliary circle of Σ' is the circle through the feet of the perpendiculars from A (and A_1) on the tangents to Σ' . Thus, the projections from A on the lines p, q, r, s all lie on a circle coaxal with the pedal circles of X, Y, Z . Similar results hold good with regard to B and C . We are thus led to conclude that:

THEOREM V:—*The projections of each vertex of a triangle on the sides of an isogonal quadrilateral lie on a circle, and the three circles corresponding to the vertices belong to a coaxal system.*

THE MIQUEL-CLIFFORD CONFIGURATION

BY

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1. The main object of this paper is to prove two theorems, the one defining the cyclic or the bicircular quartic as a certain locus and the other revealing certain symmetric properties of the Miquel-Clifford configuration.

2. It is well known that

the locus of a point P which is such that the feet of the perpendiculars from P on four straight lines l_i in a plane lie on a circle is a cubic curve Γ passing through the 8 points of the Miquel-Clifford configuration generated by the four lines l_i (2.1)

If the cubic curve Γ be regarded as lying in the projective plane, it passes through the two circular points at infinity and is then familiarly known as the circular cubic. If P be a point on Γ , the feet of the perpendiculars from P on the lines l_i lie on a circle. Hence the reflections of P about the four lines l_i lie also on a circle. Now the four lines l_i are, from the standpoint of inversive

geometry, four circles passing through the ideal point at infinity Ω of the inversive plane. Since the inverse of Γ is a circular cubic or a cyclic according as the centre of inversion does or does not lie on Γ , we have by inverting the theorem (2.1)

The locus of a point P which is such that the inverse of P w. r. t. four concurrent circles lie on a circle is a cyclic curve Γ' passing through the 8 points of the Miquel-Clifford configuration generated by the four concurrent circles. ... (2.2)

It is readily seen that Γ' passes through the limiting points of the six coaxal systems of circles defined by pairs of the given set of circles. The cyclic Γ' when regarded as lying in the projective plane is a circular cubic or a bicircular quartic according as the centres of the four given circles do or do not lie on a circle. In the inversive plane Γ' passes through Ω in the former case, while in the latter it does not.

3. The configuration generated by four lines in a plane is such that the centres of the four circles concurrent at any of its points lie on a circle—the centre circle of the four circles. The centre circle is a proper circle only in the case of the four circles passing through the Miquel point of the four lines, and it reduces to the point at ∞ , Ω in the case of the four circles through Ω i.e. the four lines generating the configuration. At every other point of the configuration two of the four circles passing through the point are proper circles while the remaining two are the line circles through the point. Hence the centres of the circles through such a point lie on a line circle i.e. the line joining the centres of the two proper circles through the point. Hence by theorem (2.2) the locus Γ' associated with any tetrad of concurrent circles of the configuration is a cubic passing through the 8 points of the configuration, and when it is regarded as lying in the projective plane it passes through the circular points at infinity. Thus the curve Γ' is the same with whatever concurrent tetrad of the configuration we may start. Hence, inverting we have the theorem, that if the inverses of a point P w.r.t. a concurrent tetrad of circles of a Miquel-Clifford configuration lie on a circle then P is in the same relation with any other concurrent tetrad of circles of the configuration. Taking now a fifth circle through a point C of multiple incidence of the above configuration, if the inverses of a point P w.r.t. the five circles through C lie on a circle, it is easy to see by repeated application of the previous theorem, that the relation of P with any other concurrent

set of 5 circles of the configuration generated by the five given circles is the same; and generally we shall have the theorem:—

If the inverses of a point P w.r.t. a set of n concurrent circles of a Miquel-Clifford configuration consisting of 2^{n-1} circles and 2^{n-1} points lie on a circle, then the relation of P with any other concurrent set of n circles of the configuration is the same. ... (3.1)

The particular case of the preceding theorem obtained by taking P to be the point at infinity Ω is interesting and may be stated:

If the centres of a set of n concurrent circles of a Miquel-Clifford configuration consisting of 2^{n-1} circles and 2^{n-1} points lie on a circle then the centres of any other set of n concurrent circles of the configuration lie also on a circle.

In fact if the inverses of a point P w.r.t. the n circles of the configuration lie on a circle, we can obtain another Miquel-Clifford configuration from the given configuration by associating with each circle of the given configuration a point—namely the inverse of P w.r.t. that circle and with each point of the configuration a circle—namely the circle through the n points associated with the n circles concurrent at the point. That the set of 2^{n-1} points and 2^{n-1} circles thus obtained form a Miquel-Clifford configuration is a consequence of theorem (3.1).

It is well known that every Miquel-Clifford configuration consisting of 8 circles and 8 points is invariant under a Möbius involution†. Hence it follows that the cyclic Γ' associated with the configuration is also invariant in the same Möbius involution. But it is known that every cyclic is automorphic with respect to three mutually harmonic involutions defined by the four mutually orthogonal circles of inversion of the cyclic. Hence it follows that

“The Möbius involution under which a Miquel-Clifford configuration is invariant is one of the three mutually harmonic involutions w. r. t. which the cyclic Γ' associated with the configuration is automorphic”.

† Ramamurthi, “On a cubic transformation in circle-geometry” *Jour. Annamalai University*, Vol. III, No. 2.

A PROPERTY OF THE MATHEMATICAL EXPECTATION

BY

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1. While working out a few of Chrystal's problems on the mathematical expectation of a drawing from a bag containing coins of two kinds we noticed that the manner of drawing was not specified and therefore we worked by the two different methods that were open to us. First we assumed that the coins were drawn one by one and replaced after every attempt and alternatively that they were drawn one by one but not replaced, or what comes to the same thing, that they were drawn at the same time. Both the methods led to the same result and we were tempted to investigate whether this is true in all cases. This note gives the details of our method of attack and draws attention to another surprising but very useful result.

2. Let a bag contain a coins of one kind and b of another, each of the former being r times the latter unit in value. If k coins are drawn it is required to find the mathematical expectation of the value of the drawing.

i. *Suppose the coins are drawn one by one and replaced.*

If $\frac{a}{a+b} = p$, and $\frac{b}{a+b} = q$ the expectation in terms of the second unit is easily seen to be

$$\begin{aligned} & p^k \cdot kr + {}_kC_1 \cdot p^{k-1}q [(k-1)r+1] \\ & \quad + {}_kC_2 \cdot p^{k-2}q^2 [(k-2)r+2] + \dots \text{to } k+1 \text{ terms.} \\ & = kr [p^k + {}_kC_1 p^{k-1}q + {}_kC_2 p^{k-2}q^2 + \dots \text{to } k+1 \text{ terms}] \\ & \quad - (r-1) [{}_kC_1 \cdot p^{k-1}q + 2 \cdot {}_kC_2 p^{k-2}q^2 + \dots \text{to } k \text{ terms}]. \\ & = kr(p+q)^k - (r-1)kq(p+q)^{k-1} \\ & = kr - (r-1) \frac{kb}{a+b} = \frac{k(ar+b)}{a+b} \dots \dots \dots \text{(A)} \end{aligned}$$

ii. *Suppose the coins are drawn one by one but not replaced*

(a) If $k < a$, $k < b$ the expectation will be given by

$$\frac{{}_aC_k}{{}_{(a+b)}C_k} \cdot kr + \frac{{}_aC_{k-1} \cdot {}_bC_1}{{}_{(a+b)}C_k} [(k-1)r+1] + \dots \text{to } (k+1) \text{ terms}$$

$$\begin{aligned}
&= \frac{kr}{(a+b)C_k} \left[{}_aC_k + {}_aC_{k-1} \cdot {}_bC_1 + \dots \dots (k+1) \text{ terms} \right] \\
&- \frac{r-1}{(a+b)C_k} \left[{}_aC_{k-1} \cdot {}_bC_1 + 2 \cdot {}_aC_{k-2} \cdot {}_bC_2 + \dots \dots k \text{ terms} \right] \\
&= kr - \frac{(r-1)b}{(a+b)C_k} \left[{}_aC_{k-1} + {}_aC_{k-2} \cdot {}_{b-1}C_1 + \dots \dots k \text{ terms} \right] \\
&= kr - \frac{(r-1)b}{(a+b)C_k} \cdot {}_{(a+b-1)}C_{k-1} \text{ by Vandermonde's theorem} \\
&= rk - (r-1)b \cdot \frac{h}{a+b} - \frac{h(r(a+b))}{a+b} \dots \dots (B) \\
(\beta) \quad &\underline{a < k < b}
\end{aligned}$$

Now the expectation will be

$$\begin{aligned}
&\frac{{}_aC_a \cdot {}_bC_{k-a}}{(a+b)C_k} \left[ar + (k-a) \right] + \frac{{}_aC_{a-1} \cdot {}_bC_{k-a+1}}{(a+b)C_k} \left[(a-1)r + (k-a+1) \right] \\
&\quad + \dots \dots (a+1) \text{ terms} \\
&= \frac{(ar+k-a)}{(a+b)C_k} \left[{}_aC_a \cdot {}_bC_{k-a} + {}_aC_{a-1} \cdot {}_bC_{k-a+1} + \dots \dots \text{to } (a+1) \text{ terms} \right] \\
&- \frac{r-1}{(a+b)C_k} \left[{}_aC_1 \cdot {}_bC_{k-a+1} + 2 \cdot {}_aC_2 \cdot {}_bC_{k-a+2} + \dots \dots a \text{ terms.} \right]
\end{aligned}$$

By collecting the coefficients of x^k in the product of $(1+x)^a$ and $(1+x)^b$ we find

$${}_aC_a \cdot {}_bC_{k-a} + {}_aC_{a-1} \cdot {}_bC_{k-a+1} + \dots \dots \text{to } (a+1) \text{ terms} = (a+b)C_k,$$

Again

$$\begin{aligned}
&{}_aC_1 \cdot {}_bC_{k-a+1} + 2 \cdot {}_aC_2 \cdot {}_bC_{k-a+2} + \dots \dots \text{to } a \text{ terms} \\
&= a [{}_bC_{k-a+1} + {}_{(a-1)}C_1 \cdot {}_bC_{k-a+2} + \dots \dots a \text{ terms}] \\
&= a [{}_aC_{a-1} \cdot {}_bC_{k-a+1} + {}_{a-1}C_{a-2} \cdot {}_bC_{k-a+2} + \dots \dots a \text{ terms}] \\
&= a [{}_{(a+b-1)}C_k] \text{ arguing as before.}
\end{aligned}$$

Hence the expectation in this case is

$$\begin{aligned}
&(ar+k-a) - \frac{(r-1)a \cdot (a+b-h)}{(a+b)} \\
&= ar+k-a - (r-1)a + \frac{ka(r-1)}{a+b} \\
&= \frac{k(ra+b)}{a+b} \dots \dots (c)
\end{aligned}$$

If $a > k > b$ the result will be the same as can be seen by changing r into $\frac{1}{r}$, multiplying the result by r and then interchanging a and b .

$$(\gamma) \quad \underline{k > a > b}$$

Now the expectation is

$$\begin{aligned} & \frac{aC_a \cdot bC_{k-a}}{a+bC_k} \cdot ar + \frac{aC_{a-1} \cdot bC_{k-a+1}}{(a+b)C_k} \left[(a-1)r + (k-a+1) \right] \\ & \quad + \dots + \frac{aC_{k-b} \cdot bC_b}{(a+b)C_k} \left[(k-b)r + b \right] \\ & = \frac{(ar+k-a)}{(a+b)C_k} \left[aC_a \cdot bC_{k-a} + aC_{a-1} \cdot bC_{k-a+1} + \dots + aC_{k-b} \cdot bC_b \right] \\ & = \frac{r-1}{(a+b)C_k} \left[aC_{a-1} \cdot bC_{k-a+1} + 2aC_{a-2} \cdot bC_{k-a+2} + \dots \text{to } (a+b-k) \text{ terms} \right]. \end{aligned}$$

This reduces to

$$\begin{aligned} & ar + k - a - \frac{(r-1)a \cdot (a+b-1)C}{(a+b)C_k} \\ & = \frac{k(ra+b)}{a+b} \dots \dots \dots (D) \end{aligned}$$

3. The mathematical expectation is thus independent of the manner of drawing. It will also be seen that in all cases *it is equal to the average value of a coin in the bag multiplied by the number of coins drawn*. It may be remarked that the application of this proposition will save much labour that is now being wasted in the tedious process of calculating the individual probabilities for each case and then finding the weighted average.

4. Example.

A bag contains 3 sovereigns and 8 shillings. A man has the option first of drawing two coins at once or secondly, one coin and afterwards another provided the first be a shilling. Which had he better do?

The usual method

In the first option the expectation is

$$\begin{aligned} & \frac{3C_2}{11C_2} \times 40 + \frac{3C_1 \cdot 8C_1}{11C_2} \times 21 + \frac{8C_2}{11C_2} \times 2 \text{ sh.} \\ & = \frac{40 \times 3}{55} + \frac{24 \times 21}{55} + \frac{28 \cdot 2}{55} = \frac{680}{55} = 12 \frac{4}{11} \text{ sh.} \end{aligned}$$

In the second option it is

$$\begin{aligned} & \frac{3}{11} \times 20 + \frac{8}{11} \cdot \frac{3}{10} \times 21 + \frac{8 \times 7}{11 \times 10} \times 2 = \frac{60}{11} + \frac{252}{55} + \frac{56}{55} \text{ sh.} \\ & = \frac{608}{55} \text{ sh.} = 11 \frac{3}{55} \text{ sh.} \therefore \text{The first is better.} \end{aligned}$$

The New Method

The expectation in the first draw = $\left[\frac{68}{11} \times 2 \right] \text{ sh.} = 12 \frac{4}{11} \text{ sh.}$

$$\begin{aligned} \text{,, ,, second draw} &= \frac{68}{11} + \frac{8}{11} \times \frac{67}{10} \text{ sh.} \\ &= \frac{680 + 536}{110} = \frac{1216}{110} \text{ sh.} \end{aligned}$$

Thus the first is better. $= 11 \frac{3}{55} \text{ sh.}$

N. B.—It could also be argued that the expectation of a draw of two coins simultaneously is the same as that of a draw of two, one by one without restriction. But as there is a restriction in the second option the expectation will be less. Hence the first is better.

This paper has been concerned only with the case of two kinds of coins and the method followed has been of the traditional type. In a paper to be published shortly the most general problem concerning any number of different kinds of coins is solved, but here the usual method fails and a distinctly new approach has had to be made.

GLEANINGS

The number, $M_{257} = 2^{257} - 1 =$

231584, 178474, 632390, 847141, 970017, 375815, 706539, 969331, 281128, 078915, 168015, 826259, 279871,

proved composite by Kraitchik and D. H. Lehmer, is interesting as being the largest 'genuine' composite number known, that is to say the largest number which is known to be composite but of which no factor has been found.

HARDY and WRIGHT: *An Introduction to the Theory of Numbers*, 1938.

But at this moment of exaltation Brouwer appears with something that looks suspiciously like a flaming sword in his strong right hand. The chase is on: Dedekind, in the role of Adam, and Cantor disguised as Eve at his side, are already eyeing the gate apprehensively under the stern regard of the uncompromising Dutchman. The postulational method for securing freedom from contradiction proposed by Hilbert will, says Brouwer, accomplish its end—produce no contradictions—but "nothing of mathematical value will be attained in this manner; a false theory which is not stopped by a contradiction is none the less false, just as a criminal policy unchecked by a reprimanding court is none the less criminal.

E. T. BELL: *Men of Mathematics*

ON FINITE BOOLEAN ALGEBRA*

BY

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This paper is devoted to a discussion of the known theory—namely, the representation of the finite Boolean Algebra as an algebra of Classes and as an algebra of Vectors. The ideas involved in the usual proofs of the representation (namely ‘A finite Boolean Algebra can be regarded as an algebra of all sub-classes of a given class’) seemed to the present writer to call for more clarification and precision. Hence, a systematic proof of the above theorem is attempted on the basis of the ideas of the isomorphism between groups and also between two Boolean Algebras. (See Sections IV, V of this paper). On more or less similar lines, the representation of a finite Boolean Algebra as a Vector algebra (mod. 2) is also discussed. The first section of this paper, consists of a brief account of the development and characteristic properties of the Boolean Algebra; while the second section pertains to the theory of Minimal Elements in a Boolean Algebra.

I. *The Boolean Algebra—Development and Properties.*

George Boole (in 1847) founded a symbolic method of treating the ‘Algebra of Logic’,—also called Boolean Algebra after him. According to its laws, we are able to solve problems of logic by symbolic methods as in pure mathematics. Subsequently after Boole, other logicians chiefly Pierce and Schröder developed what has been called the Classic or Boole-Schröder algebra of logic. This was primarily applied to the abstract system interpreted as the Calculus of Classes which sets forth the possible relations between classes and inferences from class-relations. Other interpretations are also possible for the Boole-Schröder algebra, e.g. in terms of propositions. Fundamentally, propositions are more important than classes and in the *Principia Mathematica*, the algebra of Propositions is dealt with first and then systematically the algebra of classes is deduced from what are known as propositional functions. In Logic, the algebra of

* This paper formed part of a thesis for which the degree of M.Sc. was awarded by the University of Madras.

I take this opportunity to thank Dr. Vaidyanathaswamy for his kind guidance during the preparation of this paper.

classes has special importance. For, even in Boole's time, it was known that the conception of truth and falsehood could be formally expressed as an isomorphism between the propositional calculus and the two-element algebra of classes.

The term Boolean Algebra is a more modern one and has come to be accepted as denoting an algebra which has the same formal type as an algebra of all sub-classes of a given class. The calculus of propositions though in general features similar to a Boolean Algebra, yet is not a Boolean Algebra; for there are no unique propositions which correspond to the 'Zero' and the 'Universe' of the Boolean Algebra.

A Boolean Algebra can be roughly defined as a set of elements ($a, b, c, \dots$) with two binary operations $+$ and $\times$, and a unary operation called negation satisfying the following properties:

[Various systems of postulates for the class-calculus have been proposed by E. V. Huntington (5), B.A. Bernstein (1, 2), M. H. Stone (11) etc, with discussions as to their independence, consistency and so on. The earliest of these (which is also according to Eaton (3), Henle and Chapman (4) the best of all) is due to E. V. Huntington* (5). However, since in the present paper we are not concerned with the axiomatic point of view, we shall set down, the important properties of the Boolean Algebra without regard to the question of their independence, consistency and so on.]

- (i) $a+b, a.b, a'$ are also elements of the Boolean Algebra.
- (ii) Associative and Commutative property of Addition.
 $a+b+c=(a+b)+c=a+(b+c); a+b=b+a.$
- (iii) Associative and Commutative property of Multiplication.
 $a.b.c=(a.b).c=a.(b.c); a.b=b.a.$
- (iv) Double Distributive Law for Addition and Multiplication.
 $a(b+c)=a.b+a.c; (a+b).(a+c)=a+b.c.$
- (v) Tautology Laws.
 $a+a=a; a.a=a.$
- (vi) Existence of the unique elements Zero (0) and Universe (I) with the properties: $a+0=a; a.0=0; a+I=I, a.I=a.$
- (vii) Law of Absorption.
 $a+a.b=a.$

* The system referred to here, is the first of the three sets given.

(viii) The negative a' (of a) has the property that, to each a there exists a unique a' such that $a + a' = I$; $a \cdot a' = 0$

(ix) Laws of Duality.

$$a + b = (a' \cdot b')'; \quad a \cdot b = (a' + b')'. \quad \text{If } a = b, \text{ then } (a')' = a'' = a.$$

Definition 1. An element a is said to 'contain' an element b ($a > b$ in symbols) if an x exists such that $a = b + x$; b is said to be 'contained' in a ($b < a$ in symbols).

An immediate consequence of this definition is that

$$\text{If } a > b \text{ and } b > c, \text{ then } a > c.$$

(for, $a = b + x_1 = c + x_2 + x_1$; i.e. $a > c$) The following well-known theorems used in the course of this paper, are derived from the above definition and properties.

THEOREM 1. (a) $a + b$ is the unique element containing a and b such that it is contained in any element containing a and b .

(b) $a \cdot b$ is the unique element contained in a and b , such that it contains any element contained in a and b .

(In particular, it follows that if $D > a$, $D > b$, then $D > a + b$; and if $D < a$, $D < b$, then $D < a \cdot b$).

Proof: (a) Let D be an element such that $D > a$, $D > b$. Then by definition 1, $D = D + D = a + x_1 + b + x_2 = a + b + x_1 + x_2$. Thus $D > a + b$, i.e. $a + b$ is contained in an element D which contains a and b , and obviously, $a + b > a$, $a + b > b$.

(b) Similarly, let D be an element such that $D < a$, $D < b$. Then by definition 1, $a = D + x_1$, $b = D + x_2$. Hence, $a \cdot b = (D + x_1) \cdot (D + x_2) = D \cdot D + D \cdot x_2 + D \cdot x_1 + x_1 \cdot x_2$ (by v) $= D + x_1 \cdot x_2$ (from the properties (i)-(vii): i.e. $D < a \cdot b$.) Thus, $a \cdot b$ contains an element D contained in a and b and obviously, $a \cdot b < a$, $a \cdot b < b$.

It is easy to show that elements with these properties are evidently unique.

THEOREM 2. $I > a > 0$.

(Since by (vi) $a + 0 = a$, for all a , $a > 0$. And since

$$a + I = I, \quad I > a. \quad \text{Hence, } I > a > 0.)$$

THEOREM 3. If $a > b$, then $a = b + a$; $a \cdot b = b$.

(For, since $a > b$, by definition 1, $a = b + x$: therefore, $a \cdot b = b(b + x) = b + b \cdot x = b$. Again since $a > b$, $a = b + x$.

Therefore, $a + b = b + b + x = b + x = a$.)

THEOREM 4. *If a and b be any two elements then $a \cdot b < a$.*

(This follows from the law of absorption (vii).)

THEOREM 5. *If $a > b$, an unique x can be found such that the two equations $a = b + x$; $b \cdot x = 0$ are satisfied.*

Proof: Since $a > b$, (by theorem 3), $a = b + a \cdot (b + a) \cdot I$
 $= (b + a) \cdot (b + b') = b \cdot b + b \cdot b' + a \cdot b + a \cdot b' = b + a \cdot b'$ (from the properties of the Boolean Algebra, given above). And $b \cdot a \cdot b' = a \cdot b \cdot b' = 0$. Hence, there is an x namely $a \cdot b'$ to satisfy the two given equations.

Next to prove the uniqueness of x , assume that if possible x, y are two solutions of the given equations. Then we have

$$b + x = b + y \text{ (since each } = a)$$

$$b \cdot x = b \cdot y \text{ (since each } = 0)$$

Multiplying the first equation by b' and adding the second equation to it, we have $x(b + b') = y(b + b')$; i.e. $x = y$. Hence, $a \cdot b'$ is unique.

THEOREM 6. *If $a + b = 0$, then $a = 0$, $b = 0$.*

[for, $0 = a \cdot 0 = a(a + b) = a + a \cdot b = a$. Similarly $b = 0$.]

THEOREM 7. *If $a > b$, then $a' < b'$.*

(This follows from definition 1, (ix), and theorem 3.)

In this paper, we are concerned with a Boolean Algebra of a finite number of elements.

A simple concrete illustration of a finite Boolean Algebra, is the algebra of all sub-classes of a finite class (say F) which arises when the sum and product of two sub-classes (a) and (b) are defined as the sub-classes resulting from the 'union' and the 'intersection' of (a) and (b) respectively. The Null-class and the Finite class (F) are the Zero (0) and the Universe (I) of this Boolean Algebra. The negative (a') of a class (a) is the class of all elements in I which do not belong to (a). All the properties given above for the Boolean Algebra can be directly verified for this algebra of sub-classes. It is clear that a sub-class (a) 'contains' a sub-class (b) according to the definition given above, if (b) is part of (a).

II. The Minimals* of a Finite Boolean Algebra.

The important idea of Minimals in a Boolean Algebra was first introduced by E. V. Huntington (5) under the name of Irreducible Elements. The fundamental properties of minimals are two:—

* See B.A. Bernstein (1), for properties and alternative definition of minimals.

namely any element of the Boolean Algebra can be expressed as a sum of minimals uniquely (see theorem 12) and the product of two distinct minimals is Zero (see theorem 10). Later on, B.A. Bernstein (1) developed the theory of minimals and their duals (which he calls Maximals) more systematically. Obviously Maximals have the same relation to the operation $\times$ as minimals have to the operation $+$.

The conception of minimals is important in many ways. For instance, the theory of minimals and maximals is the counterpart in the Boolean Algebra, of the known theory in the propositional calculus of the Disjunctive and Conjunctive normal forms of a general logical expression (logische ausdrücke) involving a given number of propositions ($A_1, A_2, \dots, A_n$) as dealt with by Hilbert and Ackermann (7). Again, the theory of minimals is the key to the proof of the theorem we have set out to prove in this paper (i.e. a finite Boolean Algebra can be regarded as an algebra of all sub-classes of a finite class). Similar properties true for the generalised Boolean Algebra have been proved by M. H. Stone (10) who has for the purpose generalised the theory of minimals (or atomic elements as he calls them). Below, I give a definition* of minimals and some of their characteristic properties.

Definition 2. A non-zero element of a finite Boolean Algebra B is said to be a minimal if it does not contain any element other than itself and zero.

THEOREM 8. *Every non-zero element of a finite Boolean Algebra B contains a minimal.*

Proof: Let a be an arbitrary element ($\neq 0$) of B . Then since $a \neq 0$, a is either a minimal or not a minimal. In the former case, since a contains itself, the theorem is once proved. In the latter case, it follows from definition 2, that a contains an element b different from itself and from zero. Again b can be a minimal or not a minimal. In the former case the theorem is proved since a contains a minimal b . In the latter case, it follows in the same way as before that there is an element c contained in b different from b and different from zero. Proceeding thus, since the total number of elements contained in a is finite (because B is finite) we arrive at a minimal contained in a in a finite number of steps.

* An investigation which establishes this theorem for the general Boolean Algebra, has been made by M. H. Stone (10).

** It is easy to show that this definition is identical with Bernstein's definition. See my paper 'On Symmetric Functions of n elements in a Boolean Algebra.'—*Journal of Ind. Math. Soc.* p. 4, Vol. II, No. 5.

THEOREM 9. *If a be an arbitrary element, and m be any minimal, then either $a \cdot m = 0$ or $a \cdot m = m$.*

Proof: For by theorem 4, $am < m$. But since m is a minimal, the only elements which are $< m$ are 0 and m . Hence, $am = 0$ or $am = m$.

THEOREM 10. *The product of two distinct minimals is zero.*

For, if m_1, m_2 , be two different minimals, by theorem 9 above, $m_1 \cdot m_2 = 0$ or m_1 ; $m_2 \cdot m_1 = 0$ or m_2 . But since $m_1 \neq m_2$, the two statements can be true only if $m_1 m_2 = 0$.

THEOREM 11. *The only element which is orthogonal * to all minimals is zero.*

Proof: Let $xm = 0$ for every minimal m . We shall show this is impossible unless x is zero. For, if $x \neq 0$, by theorem 9 it contains a minimal m so that $x \cdot m = 0$. This contradicts our hypothesis namely that $x \cdot m = 0$, for every minimal m . Hence, $x = 0$.

THEOREM 12. *Every element can be expressed uniquely as a sum of minimals.*

Proof: Let a be an arbitrary element ($\neq 0$), and let all the minimals of the finite Boolean Algebra B be $m_1, m_2, \dots, m_k, m_{k+1}, \dots, m_r$. If m be any one of the minimals, then from theorem 9, $a \cdot m = 0$ or $a \cdot m = m$. Hence, all the minimals can be divided into two classes such that for the minimals of the first class $a \cdot m = m$, and for the minimals of the second class $a \cdot m = 0$; i.e. the minimals of the first class are contained in a and the minimals of the second class are not contained in a . And since $a \neq 0$, there always exist by theorem 8 minimals which are contained in a . Let $m_1, \dots, m_k (k \geq 1)$ be the minimals contained in a , and $m_{k+1}, m_{k+2}, \dots, m_r$ be those not contained in a . Then since $a > m_1, a > m_2, \dots, a > m_k$, we have by Theorem 1 $a > m_1 + m_2 + \dots + m_k$. Hence by Theorem 5, we can always find an unique x such that

$$a = (m_1 + m_2 + \dots + m_k) + x.$$

$$\text{and, } x(m_1 + m_2 + \dots + m_k) = 0.$$

And by (iii) of Sec. I and Theorem 6, it follows from the second equation, $m_1 x = 0, m_2 x = 0, \dots, m_k x = 0$.

Next considering the minimals $(m_{k+1}, m_{k+2}, \dots, m_r)$ of the second class, since by Theorem 9 $a \cdot m_{k+1} = 0; \dots; a \cdot m_k = 0$, and also

* In a Boolean Algebra a is said to be orthogonal to b , if $a \cdot b = 0$.—M. H. Stone (10).

since $a > x$, it follows from Definition 1, (iii) of Sec. I and Theorem 6 that

$$x.m_{k+1} = 0, \quad x.m_{k+2} = 0, \quad \dots, \quad x.m_k = 0.$$

Thus we see, $x.m_1 = 0, \quad x.m_2 = 0, \quad \dots, \quad x.m_r = 0$.

Hence by Theorem 4, $x = 0$.

Therefore, $a = m_1 + m_2 + \dots + m_k =$ the sum of the minimals contained in a ; and this expression is evidently unique. For the sake of convenience, we shall refer to these minimals (i. e. the minimals contained in a) as 'minimals of a '.

THEOREM 13. *The minimals of an algebra K of all sub-classes of a finite class C , are the one-element sub-classes of C .*

For, in the algebra of all sub-classes of a finite class C , the minimals by definition 2, are those sub-classes which do not contain any sub-classes other than themselves and the Null-class. Such sub-classes are obviously the one-element sub-classes of C .

III. *The Abelian Disjunction* group constituted by the Elements of B .*

Contrary to what happens in ordinary algebra, the elements of a Boolean Algebra B do not form a group with respect to either of the operations $+$ and $\times$. But however, we can define an operation called disjunction and denoted by $\oplus$ for which the elements form a group which is abelian.

Definition 3. The disjoint of two elements a, b is given by a

$$a \oplus b = ab' + a'b.$$

[Hence, in the algebra of classes, the disjoint of two classes (a) (b) is defined as that class of elements which belong to one but not to both of the classes (a) and (b)].

Some chief properties † of disjunction are :

- (a) It is commutative and associative.
- (b) $a \oplus 0 = a$; $a \oplus a = 0$; $a \oplus I = a'$.
- (c) It distributes x .

* See B. A. Bernstein (2); See also M. H. Stone (11), and also p. 103-105. Proc. of the Nat. Acad. of Sciences 1935. But the dual operation namely conjunction, is not studied by M. H. Stone in the above two papers. See also R. Vaidyanathaswamy (12).

† For proof of the property (c) see my paper—"On Symmetric Functions etc." loc. cit. page 8, th. 2.

$$(d) \quad a + b = (a \oplus b) + a.b.$$

(for, from definition 3, $a \oplus b + a.b = a.b' + a'.b + a.b$.

$$= a(b + b') + a'b = a + b).$$

Corollary. If $ab = 0$, $a + b = a \oplus b$.

From (b) we can at once see that the identity element of the group with respect to disjunction is 0, and that each element has itself as its own inverse. Also since, $a \oplus a = 0$, each element except of course 0, is of order 2. Further, since disjunction is commutative, the group is abelian.

Now from the theory of finite abelian groups, we know that any such group possesses a set of basis-elements $i_1, i_2, \dots, i_n$; so that any element of the group can be uniquely expressed in terms of the elements i , by means of the group operation. Hence, for the finite abelian group constituted by the elements of the finite Boolean Algebra B with disjunction as group operation, there exists a basis (say) $i_1, i_2, \dots, i_n$, which we may call a **Disjunction-Basis** of B; and any element of the disjunction-group B, can be expressed as the continued disjoint of elements chosen from these i 's each i occurring possibly more than once. But since every element, and therefore every basis-element is of order 2 with respect to disjunction, it follows that each i occurs only once or not at all in the simplified expression of a as the continued disjoint of the i 's. Thus, $i_1, i_2, \dots, i_n$, being a disjunction-basis of the Boolean Algebra, any element a can be expressed as the continued disjoint of a unique sub-set of these i 's.

It is clear that a disjunction-basis of B is not unique. We shall now proceed to show that the minimals of B constitute a special disjunction-basis.

THEOREM 14. *The set of minimals of a finite Boolean Algebra (B) is a disjunction-basis of B.*

Proof; For, from the corollary of (d), it follows that the sum of a finite number of elements is equal to their continued disjoint provided the product of every two of the elements is zero. Hence, by theorem (10) the sum of a number of minimals is equivalent to their continued disjoint. And since by theorem 12 any element of B can be expressed uniquely as the sum of minimals of B, it follows that any element can be expressed uniquely as the continued disjoint of minimals of B. Thus the minimals of B constitute a Disjunction-Basis.

THEOREM 15. *The minimals of $a \cdot b$, are the minimals which belong to both a and b .*

Proof: Let $i_p, i_q, \dots, i_x, i_y, \dots$ be the minimals of a, b , respectively. Then by Theorem 14 we have,

$$a = i_p \oplus i_q \oplus i_r \oplus \dots$$

$$b = i_x \oplus i_y \oplus i_z \oplus \dots$$

$$\text{Then, } a \cdot b = (i_p \oplus i_q \oplus i_r \oplus \dots) \cdot (i_x \oplus i_y \oplus i_z \oplus \dots)$$

= the continued disjoint of those minimals which belong to both a and b (since $i_p \times i_q = 0$)

IV. Isomorphism between two Boolean Algebras.

Definition 1. Two Boolean Algebras L, L_1 are said to be isomorphic if there is a one-to-one reversible correspondence between the elements of L, L_1 which preserves the 'containing' relation (i.e. if $a, b; a_1, b_1$ be pairs of corresponding elements of L, L_1 , if $a > b$, then $a_1 > b_1$ and vice-versa).

The following theorems regarding the isomorphism of two Boolean Algebras L, L_1 , are fundamental.

THEOREM 16. *If two Boolean Algebras are isomorphic, then*

- (i) O, I of either correspond to O, I of the other.
- (ii) *Minimals of one correspond to minimals of the other.*
- (iii) *Sum and product of elements of one algebra, correspond to the sum and product of the corresponding elements of the other algebra.*
- (iv) *The negative of an element of one algebra corresponds to the negative of the corresponding element of the other.*
- (v) L, L_1 , considered as disjunction-groups are also isomorphic.

Proofs:—(i) Let a, a_1 be a set of corresponding elements of L, L_1 and suppose x, y , of L_1 correspond to $0, I$ of L . Then since by Theorem 2, $I > a > 0$ for every a , it follows from the definition of isomorphism that $y > a_1 > x$ for every a_1 . This can be true only if $x = 0$ and $y = I$.

(ii) Let m be a minimal of L and let an element m_1 of L_1 correspond to m . Suppose if possible m_1 is not a minimal. Then from definition 2, m_1 must contain an x_1 such that $x_1 \neq m_1$ and $\neq 0$. Now by (i), 0 of L corresponds to 0 of L_1 ; let x of L correspond to

x_1 of L_1 . Then from definition 4, it follows that $m > x$ such that $x \neq m$, $x \neq 0$. This contradicts our hypothesis namely that m is a minimal of L . Hence m_1 must be a minimal of L_1 .

(iii) Let $a, b; a_1, b_1$, be two pairs of corresponding elements of L, L_1 and let x of L_1 correspond to $a+b$ of L . We have to prove $x = a_1 + b_1$. Now by Theorem I, $a+b$ is the unique element $>a$ and $>b$ and contained in any element which is $>a$ and $>b$. Therefore, since L, L_1 are isomorphic, x is the unique element contained in any element which is $>a_1$ and $>b_1$ and which is itself $>a_1$ and $>b_1$ i.e. by Theorem I, $x = a_1 + b_1$. Thus, $a_1 + b_1$ corresponds to $a+b$. Similarly we can prove that $a_1 \cdot b_1$ corresponds to ab .

(iv) Let a, a_1 be a pair of corresponding elements of L, L_1 . Also let a' be the negative of a . Let us assume x of L_1 to correspond to a' . Now from (vii) of Sec. I. the equations $aa' = 0$, $a + a' = I$ hold for every a and its unique negative a' . Now since L, L_1 are isomorphic, we have from (iii) above, $a_1 \cdot x$, $a_1 + x$ corresponding to aa' , $a + a'$ respectively. And since $0, I$ of L_1 correspond to $0, I$ of L , the equations $a_1 \cdot x = 0$, $a_1 + x = I$, ($I > a$ by Theorem 2) hold for a_1, x and by Theorem 5 have a unique solution for x namely $a'_1 \cdot I = a'_1$ by (v) of Sec. I. Hence, a'_1 (the unique negative of a_1) corresponds to a' .

(v) Let $a, b; a_1, b_1$ be two pairs of corresponding elements of L, L_1 . By (iv) a'_1, b'_1 correspond to a', b' respectively. Hence from (iii) $a_1 \cdot b'_1$ corresponds to $a \cdot b'$, and $a'_1 \cdot b_1$ to $a' \cdot b$; Therefore, $a_1 \cdot b'_1 + a'_1 \cdot b_1$ corresponds to $a \cdot b' + a' \cdot b$ i.e. $a_1 \oplus b_1$ corresponds to $a \oplus b$. Thus the two disjunction-groups constituted by L, L_1 are isomorphic.

THEOREM 17: *A reversible one-to-one correspondence between two Boolean Algebras L, L_1 is a Boolean isomorphism if either*

(i) *The product of any two elements of one algebra corresponds to the product of the corresponding elements of the other,*

Or, (ii) *The sum of any two elements of one, corresponds to the sum of the corresponding elements of the other.*

Proof:—(i) Let $a, b; a_1, b_1$ be two pairs of corresponding elements of L, L_1 and let $a > b$. Then by Theorem 3, $ab = b$. Since by hypothesis $a_1 \cdot b_1$ corresponds to ab , we have $a_1 \cdot b_1 = b_1$ i.e. $a_1 > b_1$. Thus by definition 4, L, L_1 are isomorphic.

(ii) Similarly for the second case, if $a > b$, $a + b = a$. And since $a_1 + b_1$ corresponds by hypothesis to $a + b$, we have $a_1 + b_1 = a_1$, i.e. $a_1 > b_1$. Thus by definition 4, L, L_1 are isomorphic.

We may remark in connection with Theorem 16 (v) that the isomorphism of L, L_1 as disjunction-groups does not imply their isomorphism as Boolean Algebras. The condition for this is given in the next theorem.

THEOREM 18. *If a group isomorphism Γ , can be established between the disjunction-groups constituted by two Boolean Algebras L, L_1 the necessary and sufficient condition for Γ to be a Boolean isomorphism, is that the minimals of the algebras correspond in Γ .*

Proof: Theorem 16 (i) and (iv) prove the necessity of the condition. To prove the sufficiency, let us assume the minimals $i'_p, i'_q, i'_r, \dots, i'_x, i'_y, i'_z, \dots$ of L_1 correspond to the minimals $i_p, i_q, i_r, \dots, i_x, i_y, i_z, \dots$ of L in the group of isomorphism Γ . Also let $a, b; a_1, b_1$ be two pairs of corresponding elements of L, L_1 . Now from Theorem 14 any element of a finite Boolean Algebra can be expressed as the continued disjoint of the minimals of the algebras; and by the same theorem, these minimals constitute a disjunction-basis of the algebra.

$$\begin{aligned}\text{Thus } a &= i_p \oplus i_q \oplus i_r \oplus \dots \\ b &= i_x \oplus i_y \oplus i_z \oplus \dots\end{aligned}$$

And from hypothesis it is clear that the minimals of a_1, b_1 correspond to the minimals of a, b respectively. Hence the minimals common to a_1, b_1 correspond to the minimals common to a, b (by theorem 15.) Therefore it follows that $a_1 \cdot b_1$ (=the continued disjoint of the minimals common to a_1, b_1 by theorem 15.) corresponds to $a \cdot b$ (=the continued disjoint of the minimals common to a, b by the same theorem). Thus by Theorem 17, the two algebras are isomorphic.

V. *The correspondence between the Finite Boolean Algebra B and the Classes of its Minimals.*

Let $i_1, \dots, i_n$ be a disjunction-basis of B . Then by expressing a as the disjoint of the i 's (say $a = i_p \oplus i_q \oplus i_r \oplus \dots$) we see that an element of the Boolean Algebra B corresponds to a definite sub-set ($i_p, i_q, i_r, \dots$) of the i 's. Conversely, to each sub set ($i_p, i_q, i_r, \dots$) of the disjunction-basis, there corresponds the unique element defined by $a = i_p \oplus i_q \oplus i_r \oplus \dots$. Thus we have established a one-to-one reversible correspondence between the elements of B , and the sub-sets of the i 's which are the elements of the algebra K of all the sub-classes of the disjunction-basis. We shall now prove that *this one-to-one reversible correspondence between the Boolean Algebras B, K , is an isomorphism between the disjunction-groups B, K .*

Proof: Let $a = i_p \oplus i_q \oplus i_r \oplus \dots$
 $b = i_x \oplus i_y \oplus i_z \oplus \dots$

so that the elements a, b of the boolean algebra B correspond respectively to the sub-classes $(i_p, i_q, i_r, \dots)$, $(i_x, i_y, i_z, \dots)$ (i.e. elements) of the algebra K of all the sub-classes of the disjunction-basis of B . Then, $a \oplus b = (i_p \oplus i_q \oplus i_r \oplus \dots) \oplus (i_x \oplus i_y \oplus i_z \oplus \dots)$. Since $i_k \oplus i_k = 0$, the above expression for $a \oplus b$ reduces to the continued disjoint of the minimals which belong to one but not to both of a and b i.e. to the minimals which belong to the disjoint-class of the sub-classes $(i_p, i_q, i_r, \dots)$, $(i_x, i_y, i_z, \dots)$. Thus $a \oplus b$ corresponds to that sub-class of K which is the disjoint of the sub-classes $(i_p, i_q, i_r, \dots)$, $(i_x, i_y, i_z, \dots)$; i.e. the disjoint of two elements of B corresponds to the disjoint of the corresponding elements of K . Hence the result.

It is clear that in this group isomorphism, any element i_k of the disjunction-basis $i_1, i_2, \dots, i_n$, corresponds to the one-element sub-class (i_k) and *vice versa*. But as already remarked above, this correspondence will not be in general a Boolean isomorphism. The necessary and sufficient condition for this is by theorem 18, that the minimals of B , correspond to the minimals of K . Now by theorem 13 the minimals of K , are the one-element sub-classes $(i_1), (i_2), \dots, (i_k)$ of the disjunction-basis i . Hence the elements $i_1, i_2, \dots, i_n$ of B which correspond to their one-element sub-classes $(i_1), (i_2), \dots, (i_n)$ must be minimals of B .

Thus the necessary and sufficient condition for the correspondence between the elements of B and the sub-classes of a disjunction-basis to be a Boolean isomorphism, is that the elements $i_1, i_2, \dots, i_n$ be the minimals of B . Hence, by choosing the disjunction-basis as the set of minimals of B , B and K are isomorphic; hence *any finite Boolean Algebra can be represented as the algebra of all sub-classes of a finite class*,—namely the disjunction-basis constituted by the minimals of the Boolean Algebra.

Now, if the disjunction-basis contains n elements, the total number of sub-classes of the disjunction-basis is 2^n . Hence, in particular, we can conclude that the order of a finite Boolean Algebra with n minimals is 2^n .

VI. Vector Representation of the Boolean Algebra.

*It is well known that an abelian group is the direct product of the cyclical groups. And in particular, the disjunction-group of order 2^n of the Boolean Algebra, whose n basis elements are all of

* Speiser (9).

order 2 can be represented as the direct product of n cyclical groups of order 2. These cyclical groups are abstractly identical with the additive groups of residue classes (mod. 2). Hence, the disjunction-group of the Boolean Algebra is abstractly identical with the additive group of vector's $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$ where the ε 's are the residue classes (mod. 2) and hence $= 0$ or 1 (mod. 2.)

We shall now obtain in a direct manner, the representation of the Boolean Algebra as a vector algebra (mod. 2) so that the disjoint of two elements may correspond to the sum (mod. 2) of two corresponding vectors. Since we have already shown that a finite Boolean algebra can be regarded as an algebra of sub-classes of its minimals, it is sufficient to effect this representation for the algebra K of all sub-classes of a finite class of n elements $(a_1, a_2, a_3, \dots, a_n)$. Now let us make each sub-class p of K to correspond to a vector $P \equiv (e_1, e_2, \dots, e_n)$ where $e_i = 1$ or 0 according as the i^{th} element a does or does not occur in p . Clearly, these vectors P form an abelian group of order 2^n with respect to addition (mod. 2) the identity-element being the vector $(0, 0, \dots, 0)$ which corresponds to the null-class or the null-element of the Boolean Algebra. We now show that the above one-to-one correspondence between the sub-classes and the vectors (mod. 2) is a group-isomorphism i.e. the disjunction-group of all sub-classes of the finite class of n elements and the additive group (mod. 2) of the corresponding vectors are isomorphic. For, let P, Q correspond to p, q respectively. Then since by definition (3) of the disjoint of two elements, a_i occurs in $p \oplus q$ only if it occurs in either p or q but not in both, it is clear that the i^{th} component of the vector corresponding to $p \oplus q$ is the sum (mod. 2) of the i^{th} components of the vectors P, Q . Hence $p \oplus q$ corresponds to $P + Q$ and therefore, the disjunction-group of all sub-classes of a finite class of n element $(a_1, a_2, \dots, a_n)$ and the additive group of the corresponding vectors are isomorphic.

Now, it is clear that since disjunction distributes $\times$ (c of sec. III), the Boolean Algebra B forms a ring in the usual sense with respect to $\oplus$ and $\times$. If we define the product of the vectors (mod. 2) as a vector whose i^{th} component is the product of the corresponding i^{th} components of the two vectors, we can see that these vectors form a ring with respect to $+$ and $\times$. These two rings are isomorphic as will be seen from the following theorem.

THEOREM 19. *The group isomorphism between the disjunction-group of the Boolean Algebra and the additive group of vectors is also a ring isomorphism between the Boolean ring w.r.t. $\oplus$ and $\times$ and the ring of the vectors w.r.t. $+$ and $\times$.*

Proof: As before let P, Q correspond to p, q respectively. And let $\alpha_i, \beta_i, \gamma_i$ be the i^{th} components of P, Q and the vector corresponding to $p \times q$ respectively. Then from the definition of the correspondence of the vectors, it is clear that $\alpha_i = 1$ or 0 ; $\beta_i = 1$ or 0 ; $\gamma_i = 1$ or $0 \pmod{2}$ according as a_i does or does not occur in the sub-classes $p, q, p \times q$ respectively. Now, $\gamma_i = 1 \pmod{2}$ if and only if $\alpha_i = \beta_i = 1$, and $= 0$, in all other cases. For the vector corresponding to $p \times q$ can have 1 as the i^{th} component only if a_i occurs in both p and q .

Hence from the definition of the product of two vectors we have $P \times Q$ corresponding to $p \times q$. Hence by Theorem 17(i) the two rings are isomorphic.

Hence finally we have the result that a finite Boolean algebra can be exhibited as a vector algebra (mod. 2.) in such a manner that the disjoint and the product of the Boolean algebra correspond to the sum and the product (defined in the special manner indicated above), of the vector algebra.

We may observe that the Boolean Matrix Algebra considered by Mr. Narasinga Rao (8), is in substance only an algebra of vectors of n components of the type considered above: i.e. no specific notions regarding matrices as such need be involved.

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NOTES AND DISCUSSIONS

Squares in Arithmetic Progression

To solve $A^2 - B^2 = B^2 - C^2$, when we have also $A + B = D^3$,
 $B + C = E^3$.

Denote by x^2, y^2, z^2 three squares in arithmetical progression, then
 $x = (-m^2 + 2mn + n^2)\rho$; $y = (m^2 + n^2)\rho$; $z = (m^2 + 2mn - n^2)\rho$.

We have:

$$x + y = 2n(m + n)\rho; \quad y + z = 2m(m + n)\rho;$$

so that replacing m, n by $2m^3$ and $2n^3$ and taking $\rho = (m^3 + n^3)^2$
 we have:

$$A = 4(-m^6 + 2m^3n^3 + n^6)(m^3 + n^3)^2$$

$$B = 4(m^6 + n^6)(m^3 + n^3)^2$$

$$C = 4(m^6 + 2m^3n^3 - n^6)(m^3 + n^3)^2$$

$$A + B = 8n^3(m^3 + n^3)^2$$

$$B + C = 8m^3(m^3 + n^3)^2.$$

Put $m = 2, n = 1$

$$A = -15228; \quad B = 21060; \quad C = 25596.$$

$$A^2 - B^2 = B^2 - C^2 = 211631616$$

$$A + B = 18^3; \quad B + C = 36^3.$$

Put $m = 4, n = 3$. Then

$$A = 2\ 948\ 036$$

$$B = 159\ 823\ 300$$

$$C = 226\ 005\ 052$$

$$A + B = 162\ 771\ 336 = 546^3$$

$$B + C = 385\ 828\ 352 = 728^3$$

$$A^2 - B^2 = B^2 - C^2 = 25\ 534\ 796\ 306\ 632\ 704.$$

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 Germany. }

JOHANNES MAHRENHOLZ.

Lecture Notes

The Editor invites for publication under this heading brief criticisms of results and proofs in Text Books, Questions from University examination papers; and such other material as is likely to be helpful in the collegiate class room.

I. On the Convergency of the series $\sum 1/n^k$.

$$1. \text{ Let } S_n \equiv 1 + \frac{1}{2^k} + \frac{1}{3^k} + \dots + \frac{1}{n^k}.$$

Case (i) Let $k > 1$. We have now

$$S_{2n+1} = 1 + \frac{1}{3^k} + \frac{1}{5^k} + \dots + \frac{1}{(2n+1)^k} + \left(\frac{1}{2^k} + \frac{1}{4^k} + \dots + \frac{1}{(2n)^k} \right)$$

$$< 1 + \frac{1}{2^k} + \frac{1}{4^k} + \dots + \frac{1}{(2n)^k} + (\dots)$$

$$\therefore S_{2n+1} < 1 + \frac{2}{2^k} \left(1 + \frac{1}{2^k} + \frac{1}{3^k} + \dots + \frac{1}{n^k} \right)$$

$$\text{that is } S_n < S_{2n+1} < 1 + \frac{1}{2^{k-1}} S_n$$

$$\therefore S_n < 2^{k-1} / (2^{k-1} - 1)$$

Thus S_n is bounded above as $n \rightarrow \infty$. Hence the series cannot diverge and, being a positive term series, it cannot oscillate. So it is convergent.

2. Case (ii) Let $k = 1$. Then

$$S_{2n} = \frac{1}{2} + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{(2n-1)} + \left(\frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2n} \right)$$

$$> \frac{1}{2} + \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \dots + \frac{1}{2n} \right) + (\dots)$$

$$\therefore S_{2n} > \frac{1}{2} + 2 \left(\frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2n} \right) = \frac{1}{2} + S_n$$

By repeated applications the sum of 2^r terms is seen to be greater than $\frac{1}{2} r$ and hence the series diverges.

3. Case (iii) Let $k < 1$.

This is easily seen to be divergent as each term is greater than the corresponding terms of $\sum 1/n$.

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Parlakimidi, }

P. V. L. NARASINGA RAO.

II. To determine whether a given point is in the acute or obtuse angle between two given lines.

The following is a simple method.

Let P be the point, O the intersection of the lines. Draw PM, PN perpendiculars to each of the lines. P lies in the acute or obtuse angle between the lines according as

$$PM^2 + PN^2 < \text{or} > MN^2$$

If the lines are taken in the form

$$L_i (x, y) \equiv p_i - x \cos \alpha_i - y \sin \alpha_i = 0 \quad (i=1, 2)$$

then the point (x', y') lies in the acute or obtuse angle between the lines according as

$$L_1(x', y') L_2(x', y') \cos(\alpha_2 - \alpha_1) < \text{or} > 0.$$

Lahore }

F. C. AULUCK.

Prof. Hukam Chand of S. D. College, Lahore, discusses the same problem taking the general equation of the lines and obtains the condition

$$(l_1x' + m_1y' + n_1)(l_2x' + m_2y' + n_2)(l_1l_2 + m_1m_2) < \text{or} > 0.$$

For a treatment of a similar problem regarding bisectors vide: *The Math. Student* Vol. III, p. 105.

III. Trigonometric Formulae.

In this note the well-known Trigonometric formulae for the sum or difference of two angles are obtained from the same figure which is simpler than those usually used for the purpose.

In the triangle ABC let AD be perpendicular to BC and CE perpendicular to BA produced.

1. $BA \cdot CE = 2 \times \text{area of the triangle } ABC = BC \cdot AD$
 $= DC \cdot AD + BD \cdot AD.$

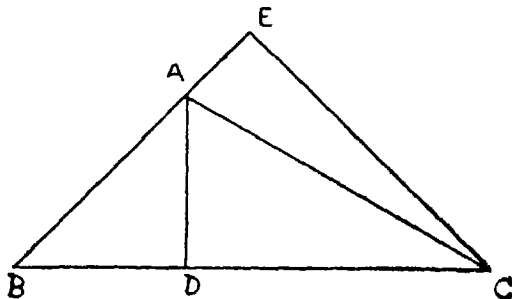


FIG. 1

Dividing both sides by $BA \cdot AC$, we get

$$\sin(B+C) = \sin CAE = \sin B \cdot \cos C + \cos B \sin C$$

$$2. AD^2 = AB^2 - BD^2$$

$$= BA (BE - AE) - BD (BC - DC) \\ = BD \cdot DC - BA \cdot AE \quad [\because BA \cdot BE = BD \cdot BC]$$

Dividing both sides by $BA \cdot AC$, we get

$$\sin B \sin C = \cos B \cos C - \cos (B + C) \\ \text{or } \cos (B + C) = \cos B \cos C - \sin B \sin C$$

$$3. \sin ACB = \frac{AD}{AC} = \frac{BC \cdot AD}{BC \cdot AC} = \frac{BA \cdot EC}{BC \cdot AC} \\ = \frac{BE \cdot EC - AE \cdot EC}{BC \cdot AC}$$

$$= \cos EBC \sin EAC - \cos EAC \sin EBC.$$

$\angle ACB = A - B$, if $\angle EAC = A$ and $\angle EBC = B$.

$$\therefore \sin (A - B) = \sin A \cos B - \cos A \sin B.$$

$$4. \cos ACB = \frac{CD \cdot BC}{AC \cdot BC} = \frac{BC^2 - BD \cdot BC}{AC \cdot BC} = \frac{BC^2 - BA \cdot BE}{AC \cdot BC} \\ = \frac{EC^2 + BE^2 - BA \cdot BE}{AC \cdot BC} = \frac{EC^2 + BE \cdot AE}{AC \cdot BC}$$

$$= \sin CAE \sin CBE + \cos CAE \cos CBE$$

$$\therefore \cos (A - B) = \sin A \sin B + \cos A \cos B.$$

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P. KESAVA MENON,
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IV. On Stationary positions of planets.

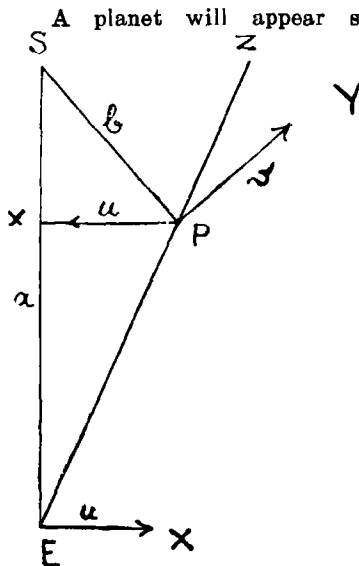


FIG. 2

A planet will appear stationary to an observer when its representative point on his celestial sphere has no velocity; in other words when the actual velocity of the planet in space is along the line of sight either way. The determination of these stationary positions as given by Ball and Godfrey are somewhat long as they ignore the kinematical aspect of the question. Barlow and Bryan obtain equation (A) from simple geometrical considerations but have not proceeded further. The following purely kinematical treatment though straight forward, may not be out of place under "Lecture Notes"

Let S be the Sun, E the Earth and P a Planet. It is assumed usual, that the two orbits are as circular and coplanar.

Let 'u' be the velocity of E, 'v' that of P. Their directions are perpendicular to SE and SP respectively, and are indicated in the figure by EX and PY.

Let $SE=a$, $SP=b$, and Z a point, on EP produced.

Now we have to express the condition that the velocity of P relative to E is along EP or PE; hence the component of this relative velocity perpendicular to EP should vanish.

For this purpose, reduce E to rest by imposing on P a velocity equal and opposite to 'u', i.e., in the direction PX' where PX' is parallel to EX. Now P has two velocities, 'u' and 'v'. Their component perpendicular to EP is

$$\begin{aligned} & u. \sin EPX' - v. \sin YPZ \\ &= u. \sin PEX - v. \cos SPZ \\ &= u. \cos E + v. \cos P \quad (\text{where E and P stand for} \\ & \qquad \qquad \qquad \angle SEP \text{ and } \angle SPE.) \end{aligned}$$

$$\text{Thus } u \cos E + v \cos P = 0 \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (A)$$

Now we proceed by the usual method, viz.,

$$\begin{aligned} a &= b. \cos \theta + PE. \cos E. \\ b &= a. \cos \theta + PE. \cos. P. \end{aligned}$$

$$\begin{aligned} \text{Hence, } \frac{a - b. \cos \theta}{b - a \cos \theta} &= \frac{\cos. E}{\cos P} = - \frac{v}{u} \text{ from (A)} \\ &= - \frac{b^{-1/2}}{a^{-1/2}} = - \frac{\sqrt{a}}{\sqrt{b}} \end{aligned}$$

$$\sqrt{a} (b - a \cos. \theta) + \sqrt{b} (a - b \cos \theta) = 0.$$

$$\therefore \cos \theta = \frac{(\sqrt{a} + \sqrt{b}) \sqrt{ab}}{a\sqrt{a} + b\sqrt{b}} = \frac{\sqrt{ab}}{a + b - \sqrt{ab}}$$

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The figure shows that trapezium $ARSB = \frac{b-a}{2} \{f(a) + f(b)\}$;

$\therefore |J| \equiv$ just the area between the curve AB and the chord AB and this is less than the area of the triangle KAB .

Now, $\Delta KAB = \Delta AFT - \Delta AFB - \Delta KTB$.

$$\begin{aligned} &= \frac{1}{2} AF \cdot FT - \frac{1}{2} AF \cdot FB - \frac{1}{2} TB \cdot \frac{1}{2} TB \tan KTB \\ &= \frac{1}{2} (b-a)^2 \tan \alpha - \frac{b-a}{2} [f(b) - f(a)] - \frac{TB^2}{4} \cot \alpha \\ &= \frac{(b-a)^2 \tan \alpha}{2} - \frac{b-a}{2} \{f(b) - f(a)\} \\ &\quad - \frac{1}{4} [(b-a) \tan \alpha - \{f(b) - f(a)\}]^2 \cot \alpha \\ &= \frac{(b-a)^2}{4} \tan \alpha - \frac{\{f(b) - f(a)\}^2}{4} \cot \alpha \\ &= \frac{(b-a)^2}{4} M - \frac{1}{4M} \{f(b) - f(a)\}^2. \end{aligned}$$

3° To prove that

If $|\phi''(x)| \leq M$ in $(a \leq x \leq b)$ and $\phi(a) = \phi(b) = 0$

then $|\phi(x)| \leq \frac{M(b-a)^2}{8}$.

Proof.—Write $\phi(x) = \int_a^x f(x) dx$.

so that the problem becomes:—

If $|f'(x)| \leq M$ in $(a \leq x \leq b)$ and $\int_a^b f(x) dx = 0$

then $\left| \int_a^x f(x) dx \right| < \frac{M(b-a)^2}{8}$.

[Note:—There is no condition that $f(a) = f(b) = 0$]

$\therefore \int_a^b f(x) dx = 0$, the curve $y = f(x)$ must cross the x -axis

at least once in (a, b) say at L , so that the positive area and the negative area exactly, balance each other. Let R, S be the points $(a, 0)$ and $(b, 0)$ and let A, B be the points $\{a, f(a)\}$ and $\{b, f(b)\}$ respectively. Since $|f'(x)| < M$ it is possible to draw a line PLQ making an angle $\tan^{-1} M$ with RS and meeting the ordinates $x=a$ and $x=b$ at P and Q so that the two areas are included in the triangles PRL and LSQ .

Now $\left| \int_a^x f(x) dx \right| \leq \text{area under } AL = \text{area under } BL \text{ numerically.}$

And obviously

area under AL $< \Delta$ PRL,

area under BL $< \Delta$ QSL.

$$\therefore \left| \int_a^x f(x) dx \right| < \text{smaller of the two triangles.}$$

i.e. $< \Delta$ QSL.

Now, the *smaller* triangle QSL must have its base $SL \leq \frac{1}{2}(b-a)$.

$$\therefore SQ < \frac{1}{2}(b-a)M.$$

$$\therefore \Delta QSL < \frac{(b-a)^2}{8}M.$$

$$\therefore \left| \int_a^x f(x) dx \right| < \frac{(b-a)^2}{8} M.$$

4° To prove that

$$\left| f(x) - f(a) - \frac{x-a}{b-a} \{f(b) - f(a)\} \right| < \frac{M(b-a)^2}{8}$$

where $f''(x) \leq M$.

Notice that if we write $|\phi(x)|$ for the left hand side then $\phi''(x)$ is exactly $f''(x)$. Thus this example is exactly the last one in 3° in a disguised form.

5° If $|\phi''(x)| \leq M$ in $a \leq x \leq b$ and $\phi(a) = \phi(b) = 0$, and $\phi'(a) = \phi'(b) = 0$ then

$$|\phi(x)| < \frac{M(b-a)^2}{16}.$$

Proof.—Write $\phi(x) = \int_a^x f(x) dx$ so that we have to prove that

$$\left| \int_a^x f(x) dx \right| < \frac{M(b-a)^2}{16}, \quad f'(x) < M; \text{ it being given that } f(a) = f(b) = 0$$

$$\text{and } \int_a^b f(x) dx = 0.$$

The condition implies that the area above the x -axis is equal to that below it.

[NOTE.—The point P in Fig. 2 where the curve crosses AB need not be the middle point of AB, though it appears as such.]

The area under the curve AP is equal and opposite to that under the curve PB,—one being positive, the other negative.

Obviously $\int_a^x f(x) dx < \text{area under AP}$. Now the area under AP $< \text{isosceles } \triangle AQP$ and the area under PB $< \text{isosceles } \triangle PRB$ and since in magnitude both areas are equal,

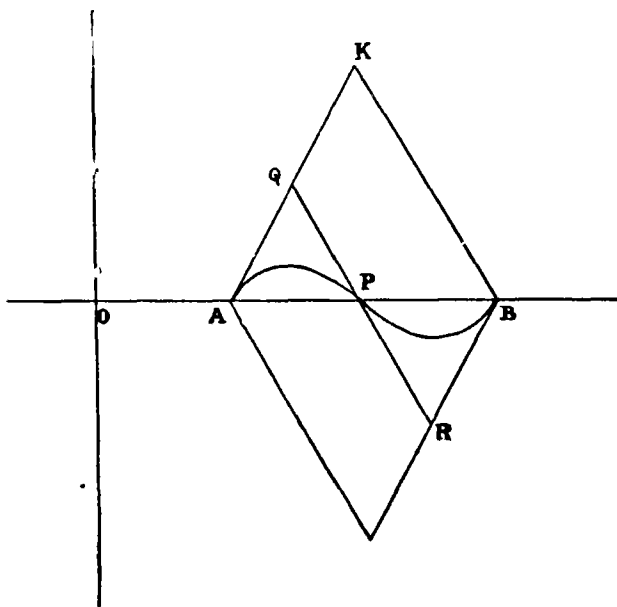


FIG. 2

$\therefore$ area under AP $<$ the smaller of the two isosceles triangles.

The base of the smaller triangle does not exceed $\frac{1}{2}$ AB.

$\therefore$ The area $< \frac{1}{8} (b-a)^2 \tan \alpha$ where $\alpha = \angle K\hat{A}B$.

Thus $\left| \int_a^x f(x) dx \right| < \frac{(b-a)^2}{16} M$.

Note that the result is true even if the curve crosses AB several times.

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G. S. MAHAJANI.

ASTRONOMICAL NOTES

Planet Notes From January to March 1939

All times given here are I. S. T. and the co-ordinates of planets given are for 5 hours 30 minutes I. S. T.

Sun. The Sun after passing the Winter Solstice on 22nd December 1938, moves with decreasing South Declination till 21st March at which time the Sun reaches the vernal equinox. The beginning of Spring indicates the passing of the Sun from South to North of the equator and it moves with an increasing North declination.

The Sun's position at the beginning of each month will be as follows :—

	<i>1st January</i>	<i>1st February</i>	<i>1st March</i>
R. A. ...	18 h 42 m	20 h 55 m	22 h 44 m
Decl. ...	23° 6'S	17° 25'S	8° 0' S.

Moon. The following phases of the Moon will occur during the period January to March 1939.

	<i>Full Moon</i>	<i>Last Quarter</i>	<i>New Moon</i>	<i>First Quarter</i>
January	6 : 3- 0 A.M.	12 : 6-40 P.M.	20 : 6-57 P.M.	28 : 8-30 P.M.
February	4 : 1-25 P.M.	11 : 9-42 A.M.	19 : 1-58 P.M.	27 : 8-56 A.M.
March	5 : 11-30 P.M.	13 : 3-7 A.M.	21 : 7-19 A.M.	28 : 5-46 P.M.

The Moon will be nearest the earth on 6th January, 4th February and 4th March and it will be farthest away on 21st January, 17th February and 16th March.

Moon's positions.

	<i>1st January</i>	<i>1st February</i>	<i>1st March</i>
R. A. ...	2 h 5 m	5 h 36 m	6 h 15 m
Decl. ...	13° 53' N.	20° 2' N.	19° 20' N.

Mercury. Mercury moves with a direct motion till 24th March when it will be stationary and changes its direction. During the early part of January the planet is visible as an evening object for about an hour after sunset and later it is too near the Sun to be seen. Mercury will be at its greatest elongation West and East on

130 ASTRONOMICAL NOTES FROM JANUARY TO MARCH 1939

3rd January and 17th March, and will be in superior conjunction with the Sun on 19th February. It will also be in conjunction with the Moon on 18th January and 22nd March.

Positions of the planet:—

	<i>1st January</i>	<i>1st February</i>	<i>1st March</i>
R. A. ...	17 h 5 m	20 h 5 m	23 h 18 m.
Decl. ..	20° 39' S	22° 0' S	5° 38' S.

Venus. During the period January to March Venus is a morning object rising about 3 hours before the Sun and its motion is direct throughout. Venus will be at its greatest elongation west on January 30th, and will be in conjunction with the Moon on 16th January, 15th February, and 17th March.

The positions of the planet will be as follows:—

	<i>1st January</i>	<i>1st February</i>	<i>1st March</i>
R. A. ...	15 h 44 m.	17 h 36 m.	19 h 46 m.
Decl. ...	15° 37' S	19° 48' S	19° 35' S.

Mars. This planet will be seen rising on the eastern sky after midnight. It rises earlier day by day and by the end of March it will rise by midnight. It will be in conjunction with the Moon on 14th January, 12th February and 13th March.

. The positions of the planet will be as follows:—

	<i>1st January</i>	<i>1st February</i>	<i>1st March</i>
R. A. ...	14 h 41 m.	15 h 58 m.	17 h 9 m.
Decl. ...	14° 42' S	19° 48' S	22° 36' S.

Jupiter. During the early part of January Jupiter will be visible as an evening object for about three hours after sunset in the western sky but later it approaches the Sun and is in conjunction with it on 6th March. It will be in conjunction with the Moon on 23rd January and 20th February.

The positions of the planet will be:—

	<i>1st January</i>	<i>1st February</i>	<i>1st March</i>
R. A. ...	22 h 11 m.	22 h 37 m.	23 h 1 m.
Decl. ...	12° 16' S	9° 50' S	7° 19' S.

Saturn. Saturn will transit about sunset on the 1st January and by the end of March, Saturn is very close to the Sun. This planet will be in conjunction with the Moon on 26th January, 23rd February and 22nd March.

The positions of the planet will be.—

	<i>1st January</i>	<i>1st February</i>	<i>1st March</i>
R. A. ...	0 h 46 m.	0 h 52 m.	1 h 2 m.
Decln. ...	2° 15' N.	3° 3' N.	4° 10' N.

Uranus. The planet will be stationary on the 22nd January and its position at the beginning of each month will be as follows :—

	<i>1st January</i>	<i>1st February</i>	<i>1st March</i>
R. A. ...	2 h 47 m.	2 h 46 m.	2 h 48 m.
Decl. ...	15° 42' N.	15° 40' N.	15° 51' N.

Neptune. Neptune will be in opposition with the Sun on 13th March and its position will be as follows :—

	<i>1st January</i>	<i>1st February</i>	<i>1st March</i>
R. A. ...	11 h 37 m.	11 h 36 m.	11 h 34 m.
Decl. ...	3° 43' N.	3 53' N.	4° 9' N.

Popular Astronomy gives an interesting announcement of the discovery of two more new satellites of Jupiter (Jupiter X and Jupiter XI). These two satellites were discovered at the Mount Wilson Observatory by Seth B. Nicholson on photographs made with the 100 inch telescope. Both are of the 19th magnitude and hence they can be seen only through powerful telescopes.

Trivandrum.

H. SUBRAMANI AIYAR.

ANNOUNCEMENTS AND NEWS

Dr. Ram Behari and Dr. R. Vaidyanathaswami have been elected Hon. Secretaries of the Indian Mathematical Society for 1938-40.

THE INTERNATIONAL CONGRESS OF MATHEMATICIANS

On invitation by the American Mathematical Society, an International Congress of Mathematicians will be held in Cambridge, Massachusetts, in 1940, the dates being September 4-12.

Former Congresses. In connection with the World's Columbian Exposition in Chicago in 1893 there was an International Congress of Mathematicians. The first congress in Europe was held at Zurich in 1897, and the most recent one met in Oslo in 1936. In the interim, with a few omissions necessitated by the World War, there have been sessions about once every four years. These have all been in Europe except for that of 1924, which was held at the University of Toronto.

It is expected that the Society will be able to furnish room and board without charge for a considerable number of foreign guests during the week that they are in Cambridge.

Following precedent, there will be a score of invited addresses, each of an hour's length, and sectional meetings for the presentation of short papers.

An innovation will be conferences, somewhat after the pattern of recent international gatherings in Moscow for Topology and in Zurich for Probability, each devoted to some field in which a vigorous advance has recently been made or is in progress. Among the conferences will be one on Algebra, with Professor A. A. Albert as chairman, one on the Theory of Measure, Probability, and Allied Topics, with Professor Norbert Wiener as chairman, one on Mathematical Logic with Professor H. B. Curry as chairman, and one on Topology with Professor Solomon Lefschetz as chairman.

Six sections are tentatively planned for the presentation of papers: (i) Algebra and Theory of Numbers; (ii) Analysis; (iii) Geometry and Topology; (iv) Probability, Statistics, Actuarial Science, Economics; (v) Mathematical Physics and Applied Mathematics; (vi) Logic, Philosophy, History, Didactics. The International Commission on the Teaching of Mathematics proposes to have a session in connection with the Congress.

Membership. Membership in the Congress will be open to all persons, whether they are able to be present in person or not. For regular members of the Congress, the fee is \$ 10.00; these persons may communicate papers and will receive the Proceedings of the Congress.

Information. Those interested in receiving information may file their names in the Office of the Society, and such persons will receive from time to time information regarding the program and arrangements

Communications should be addressed to the American Mathematical Society, 531 West 116th Street, New York City, U.S.A.

(Adapted from the Bull. of the American Mathematical Society, September, 1938)

As usual Christmas has been a period of group activities. The Indian Science Congress met in Lahore under the Presidentship of Dr. J. C. Ghosh of the Dacca University. The Mathematics and Physics section was presided over by Dr. K. R. Ramanathan of the meteorological office, Poona. His Presidential address dealt with "The Earth's Magnetism and the upper atmosphere."

The Second Indian Statistical Conference met at Lahore, with Dr. T. E. Gregory, Economic Advisor to the Government of India as President. The Presidential Address dealt with the broad problems of the relation between the Statistician and the Community.

The Indian Academy of Sciences celebrated its Fourth Session at Madras under the Presidency of Sir C. V. Raman, its President-Founder. This meeting was noteworthy as it synchronised with the Fiftieth Birthday of Sir C. V. Raman and the completion of ten years of research on the Raman Effect. An address and a special commemoration number of the Proceedings of the Academy were presented to him on the occasion. The Indian Mathematical Society of which Sir Raman is an Honorary Member joins in wishing him many long years of health and usefulness.

Mr. R. S. Varma of the Christ Church College, Cawnpore, has been awarded the degree of Doctor of Science of the Allahabad University. The subject of his thesis was "Parabolic Cylinder Functions."

Dr. Ram Behari, Reader in Mathematics, Delhi University, has been elected a Fellow of the National Academy of Sciences of India.

THE LATE PANDIT HEMRAJ

PANDIT HEMRAJ, Principal, Dyal Singh College, Lahore, was born in Daffar, a village in the Hoshiarpur District of the Panjab on the 22nd



November, 1885. His father Pt. Ganga Ram, who survives him, was a poor Brahmin, and it was under hard circumstances, therefore, that Pandit Hemraj received his education. He passed his middle school examination from the S. D. School, Hoshiarpur, in 1903, and his matriculation from the Govt. High School, Hoshiarpur, in 1905. He passed his B. A. and M. A. examinations from the D. A. V. College, Lahore with great distinction standing first in the first class in the M. A. examination in 1912. L. Chamba Mal Sud of Hoshiarpur rendered him considerable financial help during these years.

Shortly after passing his M. A., he was appointed in October 1912, an assistant Professor of Mathematics in the Dyal Singh College, Lahore. In 1916, Pandit Hemraj distinguished himself by successfully solving questions set by Professor Grace of Cambridge, who had been invited by the University of the Panjab to deliver a course of extension lectures. Panditji's work thus came to the notice of the Panjab Government and Sir John Maynard, the then Vice-Chancellor of the Panjab University, moved the University to offer Pandit Hemraj a scholarship to go abroad for higher studies. The offer was not accepted for reasons of health. By hard work Panditji rose to the Principalship of the College in 1921, and served it ably till the time of his death on the night between the 12th and the 13th November, 1938, at some minutes past midnight. In 1927, when Dr. S. S. Bhatnagar devised a Magnetic Interference Balance which has now become well-known all the world over, the Mathematical analysis of the experimental results proved a serious problem. Dr. Bhatnagar consulted several mathematicians in India and abroad. The problem was finally solved by Pandit Hemraj. Panditji's death has left a void in the academic life of the Panjab University as also in mathematical India.

Panditji was extremely simple in habits, very kind to his pupils and yet a good disciplinarian, and was in peace with every one. His life was a great lesson to all who were privileged to study at his feet. The simple village folk who knew him from his early school days as classmates or otherwise did not go to the plough when the news of his death was received. Panditji leaves behind him two sons and a small daughter besides a widow. Panditji was a prominent member of the Indian Mathematical Society and was for several years a member of the executive body. He published only a small part of his work. If his personal library and note-books are looked into, I feel sure that some work of remarkable originality may be found.

Government College, }
Hoshiarpur. }

HANSRAJ GUPTA.

BOOKS RECEIVED FOR REVIEW

- JAHNKE-EMDE: *Functionentafeln mit Formeln und Kurven*, 3rd (revised) Edition, 1938, B. G. Teubner pp. 305 Price R. M. 15 (less 25 / abroad)
- NATHAN LAZAR: *The importance of certain concepts and laws of logic for the study and teaching of geometry*, New York, 1938 pp. 66 price \$ 1.00.
- KEYSER, SMITH, KASNER, RAUTENSTRAUCH (Addresses) *Scripta Mathematica Forum Lectures* Yeshiva College, New York, 1937, pp. 94.
- G. H. HARDY and E. M. WRIGHT: *Introduction to the Theory of Numbers*, Oxford Univ. Press. 25 s.
- R. A. FRAZER, W. J. DUNCAN, and A. R. COLLAR: *Elementary matrices and some applications to Dynamics and Differential equation*: Camb. Univ. Press 1938, pp. 416.
- MILNE-THOMSON: *Theoretical Hydrodynamics*: Macmillan, 1938, pp. 552, Price 31 s. 6d.

REVIEWS

NATHAN LAZAR: *The importance of certain Concepts and Laws of Logic for the Study and Teaching of Geometry*.

The Mathematics Teacher, 525 W. 120th St. N.Y.C. 1938 pp. 66 price \$ 1.00.

This booklet deals with the logical bases of elementary theoretical geometry, the sources of study being ninety-three modern text-books besides the monumental volumes on Euclid's *Elements* by Heath and the classical works of that eminent logician, mathematician and educator, De Morgan whose strong insistence on the necessity of proper appreciation of certain logical concepts in the study of the elements of geometry did much to influence the technique of the teaching of the subject.

The author examines critically the traditional definitions of the concepts of the Converse and the Inverse and the laws of Contraposition and of Converses with special reference to elementary geometry and attempts to show that they are to a certain extent defective in statement and restrictive in scope. Alternative definitions which are wider are proposed and illustrated and their effect on the definitions of 'Necessary and sufficient Conditions' and on the methods of proof in loci problems is noticed.

Briefly the modifications suggested are as under:—To obtain the Converse of a proposition, interchange any number of Conclusions with an equal number of data. To obtain the inverse, break up the proposition into subsidiary propositions each with *one* conclusion and in each of these latter, contradict *one* of the data and also the conclusion. To obtain the contrapositive break up the proposition again as before and interchange the Contradictory of *one* of the hypotheses and that of the conclusion.

The author holds that since educational psychology points to evidence against the possibility of automatic transfer of training from one subject to another unless 'identical elements' are present in both subjects and since, transfer being a form of generalisation, training can be of such a type as to encourage or hinder such transfer, 'it is possible to utilise geometry as a medium for making the pupils conscious of the existence of logical patterns of valid and invalid reasoning in mathematics as well as in the thinking of everyday life.' He concludes with a few pedagogical recommendations and a copious bibliography of relevant literature.

Altogether, the book makes a stimulating and instructive reading and deserves careful attention with a view to ascertaining how far it would be possible to implement the suggestions in actual teaching in the High School.

Training College, }
Rajahmundry. }

K. SĀTYANARAYANA.

G. VALIRON : *Sur les Valeurs Exceptionnelles des Fonctions Mèromorphes et de leurs dérivées*, (Actualités Scientifiques et Industrielles, No. 570, price 18 frs.)

This booklet is devoted to a study of some of the recent developments in the Theory of Normal Families of Meromorphic functions. A family or class of functions Meromorphic in a domain D is said to be normal in D when any sequence of functions selected from the family contains a sub-sequence converging uniformly in any closed region interior to D . The significance of this definition will become clear if we recall that any sequence of numbers (real or complex) contains a sub-sequence converging to a limit (finite or infinite). This property does not hold, in general, for classes of elements other than numbers, even though convergence and limits could be defined for such classes. So it becomes important to consider classes of elements for which the above property is postulated. If we consider a family of functions meromorphic in a domain D and if we replace ordinary convergence of numbers by uniform convergence of functions in D and postulate for this class of functions the property specified above we are led to the idea of a normal family.

A short account of the chief criteria for normal families is found in the first chapter of the booklet under review. The connection between exceptional values and normal families is found in the fact that a family of functions regular in D is normal in D if every function of the class omits to take two specified values in D . The result is true for a family of meromorphic functions if we postulate three exceptional values. Instead of two exceptional values for the functions of a family regular in D , if we assume that no function of the class takes a specified value while the k^{th} derivatives of the functions of the family omit to take another specified value, can we say that the family is normal? The answer is in the affirmative if the exceptional value of the k^{th} ($k \geq 1$) derivatives be different from zero. A proof of this proposition is found in the second chapter. Extensions and generalisations of this last result for regular functions and corresponding results for meromorphic functions are discussed in the remaining chapters. References to original memoirs and other sources are found at appropriate places in the footnotes.

V. GANAPATI IYER.

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APPROXIMATIONS IN MATHEMATICS REGARDED AS EXACT

BY

G. A. MILLER

One of the primary facts of the history of mathematics is that approximate results were frequently regarded for long periods of time as accurate. A striking example of this fact is the use of the number 3 as the value of the ratio of the circumference of a circle to its diameter. This value was extensively employed and it is an illustration of a tendency even in mathematics to accept statements as true which could easily have been disproved. In constructing wheels for various purposes in ancient and medieval times it seems reasonable that some of the workmen observed that the given ratio exceeds 3 but their influence on the intellectual world was probably less than it is to-day, and they were less likely to be familiar with the fact that the number 3 was so widely accepted as the value of this ratio than the workmen of the present day. Hence the persistence of the number 3 for this ratio throws light on the disparity of intellectual influences in ancient times.

The fact that none of the ancient and medieval workmen are now known to have reported that the ratio of the circumference of a circle to its diameter exceeds 3, although closer approximations were also in use, appears less surprising than that the number 3 as the value of this ratio has not been expunged from all of the widely used sacred writings up to the present time. The wise admonition "Prove all things: hold fast to that which is good" was not always completely followed by the writers of our Bible even as regards mathematical questions as may be seen from the following quotation: "And he made a molten sea of ten cubits from brim to brim, round in compass, and five cubits in height thereof; and a line of thirty cubits did compass it round about." If the distance around had been actually measured it would have been found to be almost thirty-one and a half cubits instead of thirty cubits as stated in this quotation.

A large number of other approximate values for the ratio of the circumference of a circle to its diameter were employed by the ancient and the medieval mathematicians. One of these is deduced from the fact that the ancient Egyptians assumed that the area of a circle is equal to that of a square whose side is equal to $\frac{8}{9}$ th of the diameter of the circle. This is equivalent to assuming that the value of the given ratio is $256/81$ which is a much closer approximation than the number 3 and has been used as evidence tending to show that the ancient Egyptians were better mathematicians than the ancient Babylonians since the latter commonly employed the number 3 as the value of this ratio. Many other evidences, however, tend to show that the ancient Babylonians were superior to the ancient Egyptians along mathematical lines. The outstanding mathematical achievement of the latter is that they had a correct rule to find the volume of the frustum of a square pyramid but they were probably unable to prove the accuracy of this rule, judging from their other mathematical attainments.

An approximate value of the ratio of the circumference of a circle to its diameter which is closer than the ancient Egyptian value noted above, and seems to have largely superseded this value in ancient times, was given by Archimedes (287-212) in his "Measurement of the Circle." It was at first explicitly given as an approximation but was later widely assumed to be accurate and is still widely used when only rough approximations are required. Its simple form $22/7$ makes its use very convenient and it was rapidly adopted in other countries, being used in China as early as the fifth century of our era. The fact that this ratio is a transcendental number naturally prolonged the period during which various approximations to its true value were assumed to be correct since the ancient and the medieval mathematicians knew nothing about the existence of such numbers and hence were unable to explain the difficulties involved in this particular ratio. The fact that it is transcendental was first established by F. Lindemann in 1882.

The use of the number 3 for the value of the ratio of the circumference of a circle to its diameter appears also in the Koran and it is encouraging that no one is known to have been persecuted on account of his efforts to find closer approximations to the true value of this ratio. The comparatively little interest in mathematical questions on the part of the general public in ancient and medieval times is clearly exhibited by the persistence of the use of the number 3 as the value of this ratio in commonly used sacred

writings as well as by the comparatively small space devoted to this subject in the writings which have come down to us from ancient times. Only about 250 of the several hundred thousands cuneiform texts which are now in the various museums of the world are known to be devoted to mathematical tables or to actual mathematical questions, according to O. Neugebauer's "Vorlesungen über Geschichte der antiken mathematischen Wissenschaften," volume 1, 1934, page 206.

The great disparity of mathematical knowledge among the people of the same country and living at the same time is a primary fact of the history of mathematics. The study of such works as the "Elements" of Euclid, for instance, is insufficient to disclose the actual state of common mathematical knowledge among the Greeks at the time when Euclid lived. Public education has reduced the disparity of mathematical knowledge in modern times but the enormous number of the courses offered in many educational institutions in recent times tends to increase this disparity again and to lessen the amount of common knowledge even among the educated classes. The transmission of such obvious errors as the one that the value of the ratio of the circumference of a circle to its diameter is 3 in the literature which is highly recommended for its reliability is questionable since it tends to lessen the high regard for truth which should be inculcated whenever possible. Hence Christian nations may well consider the desirability of expunging this error from our Bible.

A primary fact of the history of mathematics is that square units of area and cubical units of volume were adopted in ancient times and this was the first step towards the arithmetization of geometry. Its significance was however not always realized as may be seen from the fact that the author of a geometry which appeared in Germany in about 1400 stated that he had seen many laymen who computed the area of a triangle by multiplying its base by one-half of a side which was not perpendicular to the base.* It has often been asserted that the ancient Egyptians used this inaccurate rule to find the area of a triangle but it is now probable that at least some of them know the correct rule to find this area. When the angle between the two sides is almost a right angle the half product of these two sides gives a fairly close approximation to the correct area. Since the determination of the area of a

* J. Tropicke, *Geschichte der Elementar-Mathematik*, vol. 4, (1922) p. 129.

triangle is fundamental in land measurements it is obvious that this problem attracted early attention.

Many statements which are only approximations but are not expressed as such still appear in the literature of logarithms. For instance, in such a widely used work of reference as the Weber-Wellstein "*Enzyklopädie der Elementarmathematik*" it is stated[†] that Napier's logarithm of a number is its logarithm to the base $1/e$.

Important facts in this connection are that while logarithms to the base $1/e$ do not differ in the first few decimals from those found in the tables by Napier they are essentially different since the latter were not computed with respect to any base. The statement is also very misleading since the number e was not determined until after the days of Napier. Various writers have assumed that the base of a table of logarithms could be determined by means of the numbers given for the logarithms which appear in this table. This is obviously impossible since an infinite number of bases which differ only slightly would give rise to the same logarithms as those found in such a table to a given number of decimals.

Approximations to the roots of given numbers, especially to square roots, were naturally common among the ancients since the roots of rational numbers are usually irrational and the existence of irrational quantities was first established by the ancient Greeks who, however, did not commonly assume the existence of irrational numbers. A cuneiform Babylonian text contains a rule which is the equivalent of the formula $\sqrt{a^2+b} = a + b/2a$, and rules equivalent to this approximation formula were very extensively used in the ancient mathematical literature, including that of the Greeks. In many cases a and b can be so selected that this formula gives a sufficiently close approximation for certain practical purposes. In the special case of the square root of 2 an unusually close approximation appears in the ancient Hindu literature in the following form:

$$1 + \frac{1}{3} + \frac{1}{3 \cdot 4} - \frac{1}{3 \cdot 4 \cdot 34}$$

This gives the first five decimals correctly and is one of the most remarkable approximations which is known to appear in the ancient Hindu literature, and it has given rise to an extensive literature.

One great difficulty in regard to approximate results in the history of mathematics is that it is frequently impossible to determine

[†] Vol. I, 4th ed., p. 148 (1922).

definitely whether certain results given by an author are supposed to be accurate or only approximate. A large part of mathematics is devoted to approximations which can often be carried so far that the difference between the true value and the one actually given could not be determined by the available instruments of measure. In elementary mathematics the subject of trigonometry is largely devoted to the study of approximate results. The student is thus led early to the use of approximations and it is often considered unnecessary to state explicitly that the results obtained are only approximately true. The mathematician is thus led to leave to the reader in many cases the decision whether certain results given by him are to be regarded as accurate or only approximate and various critical points are naturally obscured thereby since many readers do not like to assume responsibilities involving thinking.

Even in such simple questions as the proof of the existence of irrational quantities the mathematician insists on a degree of exactitude which can never be reached by measurement while in others, such as the use of tables of logarithms, he is satisfied with approximations which are frequently not very close. This dual character of mathematics naturally complicates its history. The ancient Greeks seems to have been the first to recognize this dual nature of mathematics and to have largely confined their attention to the exact side of mathematics, especially after their discovery of irrational quantities. In applied mathematics, on the contrary, it is commonly necessary to deal with approximations and frequently the data to which mathematical reasonings are applied are in themselves such rough approximations as to make exact methods impracticable. Hence most Greek writers failed to consider such applications.

Plato (429-348) deplored the backward state of solid geometry in his day and said that he was ashamed of it not only on his own account but also on account of all the Greeks. Fortunately, Archimedes (287-212) removed part of this shame of the Greeks by finding the beautiful formulas for the area and the volume of a sphere. However, rough approximation rules continued to be used by the Greeks even after the accurate rules became known. For instance, Heron used a rule to find the volume of the frustum of a cone by assuming that it is equivalent to that of a cylinder of the same altitude whose base is equal to the area of the section at the middle of the cone. Heron used also the correct rule to find this volume and he is supposed to have followed the ancient Egyptians in using the given approximation.

Since our Bible contains very little relating to mathematics it may be interesting to note that what may be regarded as the most astounding error in the history of mathematics, viz., the long continued and widely extended use of 3 as the value of the ratio of the circumference of a circle to its diameter, is found therein and is nowhere corrected therein. The mathematical historian is especially interested in such facts since they help him to form a satisfactory view as regards the disparity of mathematical knowledge, and the difficulty of correcting errors after they have become widely accepted as true. The writer of the present article remembers well when as a child he overheard that the given ratio is 3 and that he faithfully accepted it then as evidence of the superior knowledge of his elders who expressed no doubt as regards the accuracy of the statement, and may never have had any occasion to doubt it thereafter since they then lived in a community where only a few books were read and these were read frequently and with implicit confidence

ON THE PROPERTIES OF INTEGRAL FUNCTIONS WITH BOUNDED VALUES AT A PRESCRIBED SEQUENCE OF POINTS†

BY

V. GANAPATHI IYER

One of the important problems in the theory of Integral Functions is the determination of integral functions with specified values at a prescribed sequence of points and the investigation of the order and type of such functions. Very few significant results could be obtained without restrictions on the nature of the values assigned as well as on that of the points where the function is to take these values. When the values assigned are bounded and proper restrictions are placed on the prescribed sequence of points it is possible to obtain interesting results in this connection. In the following pages I propose to give a summary of the results and an introductory explanation of the subject-matter of the following seven papers in which I have discussed the above problem in detail.—

(A) On Integral Functions of order one and finite type (*Journal of the Ind. Math. Soc.*, New Series, Vol. 2 (1936), pp. 1-12).

† This formed the Introduction to a Thesis approved for the Degree of Doctor of Science in the Madras University.

(B) On Integral Functions of finite order bounded at a sequence of points. (*Ibid.*, pp. 53-66).

(C) Some theorems on Integral Functions bounded at a sequence of points. (*Proc. Lond. Math. Soc.*, Series 2, 43 (1937), 63-72).

(D) A note on Integral Functions of order 2 bounded at the lattice-points. (*Jour. Lond. Math. Soc.*, 11 (1936), 247-249).

(E) On Integral Functions of finite order and minimal type. (*Jour. Ind. Math. Soc.*, New Series, 2 (1936), 131-140).

(F) On the order and type of integral functions bounded at a sequence of points. (*Annals of Math.*, 38 (1937), 311-320).

(G) On effective sets of points in relation to integral functions, (*Trans. Amer. Math. Soc.*, 42 (1937), 358-365).

II. A General Account of the Papers

1. Definition and Notation. Let $F(z)$ be an integral function and $M(r, F) = \max |F(z)|$ for $|z| \leq r$. The order ρ , the lower type $\lambda(F)$, the upper type (usually known simply as the type) $k(F)$, are defined by the relations

$$\rho = \lim_{r \rightarrow \infty} \frac{\log \log M(r, F)}{\log r}; \quad \lambda(F) = \lim_{r \rightarrow \infty} \frac{\log M(r, F)}{r^\rho} \leq \overline{\lim}_{r \rightarrow \infty} \frac{\log M(r, F)}{r^\rho} = k(F).$$

When ρ is finite, $F(z)$ is said to be of finite order. For a given ρ , $F(z)$ is to be of minimal, normal or maximal type, according as $\lambda(F) = 0$, $0 < \lambda(F) < \infty$, or $k(F) = \infty$.

1.1 Let $[z_n]$ be a distinct sequence of complex numbers having their sole limit point at infinity. There is no loss of generality in supposing that the numbers $[z_n]$ are arranged according to non-decreasing moduli. We shall speak of the exponent of convergence* of $[z_n]$ as its order, for shortness. Let $\sigma(z)$ denote the canonical product with simple zeroes at $z = z_n$, $n = 1, 2, 3, \dots$. Then it is known that $\sigma(z)$ is an integral function of order ρ . Let η be the lower bound of numbers h for which

$$\sum_{n=1}^{\infty} \left| 1/\sigma'(z_n) z_n^{n+1} \right|$$

converges. We define the index of distribution (I. D.) of $[z_n]$ as max. $(0, \eta)$.

* The exponent of convergence of $[z_n]$ is defined as the lower bound of numbers h for which $\sum |z_n|^{-h}$ converges.

1.2. When z_n is real and positive for all n we denote the sequence by $[\lambda_n]$. Let $n(r)$ be the number of λ_n not exceeding r . If $\lim_{r \rightarrow \infty} \frac{n(r)}{r^\rho} = D$, we say that $[\lambda_n]$ is a measurable sequence of order ρ and density D . When $\rho=1$, the index of condensation δ of $[\lambda_n]$ is given by †

$$\delta = \overline{\lim}_{n \rightarrow \infty} \frac{1}{\lambda_n} \log \left| \frac{1}{\sigma'(\lambda_n)} \right|, \quad \sigma(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{\lambda_n^2} \right).$$

When $\rho \neq 1$, we denote by $\delta(\rho)$ the index of condensation of $[\lambda_n^\rho]$ which is of order one and density D .

2. *Papers (A) and (B)*: The problems discussed in this thesis are concerned with integral functions which take assigned values $[y_n]$ at a prescribed sequence of points $[z_n]$. Therefore it becomes necessary to consider methods for constructing functions $F(z)$ such that $F(z_n) = y_n$. The usual method is, first, to construct an integral function $g(z)$ with simple zeroes at $z = z_n$, $n=1, 2, \dots$. This is always possible by using Weierstrass's products. Then we find a sequence of integers $[p_n]$ such that the series

$$\sum_{n=1}^{\infty} \frac{y_n}{g'(z_n)} \frac{1}{z - z_n} \left(\frac{z}{z_n} \right)^{p_n}$$

converges* absolutely and uniformly for all finite z except at the points $z = z_n$. Then any integral function $F(z)$ such that $F(z_n) = y_n$ is given by

$$\frac{F(z)}{g(z)} = f(z) + \sum_{n=1}^{\infty} \frac{y_n}{g'(z_n)} \frac{1}{z - z_n} \left(\frac{z}{z_n} \right)^{p_n} \quad \dots \quad (1)$$

where $f(z)$ is an integral function to be determined from the properties of $F(z)$. When $[z_n]$ is of finite order ρ , we can take the canonical product $\sigma(z)$ for $g(z)$ in (1) and get

$$\frac{F(z)}{\sigma(z)} = F(z) + \sum_{n=1}^{\infty} \frac{y_n}{\sigma'(z_n)} \frac{1}{z - z_n} \left(\frac{z}{z_n} \right)^{p_n} \quad \dots \quad (2)$$

In (A) and (B), I have considered the problem of determining the function $f(z)$ and the sequence $[p_n]$ under suitable hypotheses. I show in theorem 5 of (A) and theorem 7 of (B) that, if $[\lambda_n]$ is a sequence satisfying the conditions

$$(i) \quad 0 < \lambda_1 < \lambda_2 < \dots < \lambda_n \rightarrow \infty;$$

† For the direct definition of the index of condensation and its properties, see V. Bernstein, *Séries de Dirichlet*, Borel Tracts (1933) 22-27. The index is never negative and is zero if $\lim_{n \rightarrow \infty} (\lambda_{n+1} - \lambda_n) > 0$.

* It is always possible to find such a sequence of integers. See, Dienes, *Taylor Series* (Oxford), 294-298.

(ii) $[\lambda_n]$ is of order ρ with $\lim_{r \rightarrow \infty} n(r) r^\rho > 0$;

(iii) $[\lambda_n^{2q}]$ is of finite I. D. where q is an integer given by $\rho \leq q < \rho + 1$;

and if $[z_n]$ denote the zeroes of

$$\sigma(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z^{2q}}{\lambda_n^{2q}}\right) \quad \dots (3)$$

then, $[p_n]$ in (2) can be replaced by a fixed integer $p > 0$ and greater than the I. D. and $f(z)$ by a polynomial of degree $p-1$ at most, provided the integral function $F(z)$ of order ρ is of sufficiently small type and is bounded at the points $[z_n]$. A similar result is discussed in theorem 4 of (B). The conclusions derived from these interpolation formulae in the papers (A) and (B) have been improved upon in the paper (C) but these formulae themselves and the method of establishing them are frequently referred to and utilised in the remaining papers.

2.1. *Paper (C)*: Here three main results are proved. In theorem (1) it is shown that when $[z_n]$ is a sequence of order ρ , $0 < \rho < \frac{1}{2}$, and finite I. D. and $\sigma(z)$, the canonical product, is such that $k(\sigma) > 0$, the hypotheses (i) $F(z)$ is of order ρ , minimal type; and (ii) $F(z_n) = O(1)$ imply that $F(z)$ is a constant. When $[z_n]$ is replaced by a measurable sequence $[\lambda_n]$ of order ρ and density D for which $\delta(\rho) = 0$ the hypotheses (i) $F(z)$ is of order ρ with $k(F) < \pi D \cot \pi \rho$; and (ii) $F(\lambda_n) = O(1)$ imply that $F(z)$ is a constant (theorem 2). Next I prove a result valid for all $\rho > 0$. Let $[\lambda_n]$ satisfy the conditions (i), (ii) and (iii) of § 2 above. Let $[z_n]$ be the zeroes of

$$\sigma(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z^{2q}}{\lambda_n^{2q}}\right).$$

If $F(z)$ is of order ρ and minimal type and if $F(z_n) = O(1)$ then $F(z)$ is a constant (theorem 3). These three are direct generalisations of the result suggested by Polya that an integral function of order one and minimal type bounded at the points $z = \pm n$, $n = 0, 1, 2, \dots$, reduces to a constant.

2.2. *Paper (D)*: Here, I prove that an integral function of order two and type less than $\frac{1}{2}\pi$ bounded at the lattice-points reduces to a constant.

2.3. *Paper (E)*: In this paper, I discuss a slightly different problem. Instead of supposing that the I. D. of the sequences considered is finite I connect the values $[y_n]$ with the sequence $[\sigma'(z_n)]$

which measures the I. D. Let $\{z_n\}$ be a sequence of order ρ , $0 < \rho < \frac{1}{2}$ and let $k(\sigma) > 0$. Let $F(z)$ be an integral function of order ρ and minimal type and let

$$X_n = \max. [|y_n|, |y_n' \sigma'(z_n)|] = O(1)$$

Then $F(z)$ is a constant.

A similar result holds for all $\rho > 0$ when $\{z_n\}$ denotes the zeros of (3) where the condition (iii) § 2 as to the finiteness of I. D. can be dispensed with. I prove two other results for functions of order one. If $[\lambda_n]$, $0 < \lambda_1 < \lambda_2 < \dots < \lambda_n \rightarrow \infty$, be such that $\lim_{r \rightarrow \infty} \frac{n(r)}{r} > 0$ and if the function $F(z)$ of order one and minimal type be such that $F(\lambda_n) = O(e^{-a\lambda_n})$ for some $a > 0$, then $F(z) \equiv 0$, the index on condensation of $[\lambda_n]$ being zero. If $F(z)$ is of order one and type less than $k(\sigma)$ where

$$\sigma(z) = O\left(1 - \frac{z^2}{\lambda_n^2}\right)$$

and if $F(\lambda_n) = O(e^{-a\lambda_n})$ however large a may be, then $F(z) \equiv 0$, the index of condensation being merely finite.

2.4. *Papers (F) and (G)*: In paper (F) I discuss results similar to that in (C) but it is assumed that for $\rho = 1$, the function in question is bounded at points lying on the four halves of the axes of reference while for $0 < \rho < 1$ they are assumed to lie on both halves of the real axis. The results obtained are correspondingly more precise. In paper (G), I consider the problem of finding a set $\{z_n\}$ for which

$$\lim_{n \rightarrow \infty} \frac{\log |F(z_n)|}{|z_n|^\rho} = k(F) = \lim_{r \rightarrow \infty} \frac{\log M(r, F)}{r^\rho}$$

where $F(z)$ is any function of order ρ and type less than d , ρ , d being two finite positive numbers. As a typical example

$$\lim_{|m+in| \rightarrow \infty} \frac{\log |F(m+in)|}{m^2+n^2} = \lim_{r \rightarrow \infty} \frac{\log M(r, F)}{r^2},$$

$m, n = 0, \pm 1, \pm 2, \dots$

for any function of order two and type less than $\pi/2$.

III. Introduction to the Thesis

1. The essential properties of a single-valued analytic function are determined by the nature and distribution of its singularities. An integral function is characterised by the fact that its sole singularity is at infinity; this must be an essential singularity unless the function reduces to a polynomial in which case the singularity is a

pole of order equal to the degree of the polynomial. Therefore an integral function might be regarded as the simplest generalisation of a polynomial.

1.1. The most characteristic property of a polynomial is that it takes any given finite value at n points, n being the degree of the polynomial. If a transcendental integral function be regarded as a polynomial of infinite degree we should expect that it assumes any finite value at an infinity of points. This is, in fact, true with the modification that we must allow for the possible existence of a single value for which this statement ceases to be true. This is the content of the famous Picard's theorem which states that a transcendental integral function takes any finite value, save possibly one, at an infinity of points.

1.2. A polynomial of degree n takes a given value exactly n times. A similar result can be stated for functions of finite order. Let $\{z_n(a)\}$ be the sequence of points where the function $F(z)$ of order ρ takes * the value a . Let $E(a)$ be the exponent of convergence of $\{z_n(a)\}$. Then $E(a) = \rho$ except possibly for a single value a for which $E(a)$ might be less than ρ . This result is due to Borel. The classical theory of integral functions is concerned with the generalisation and detailed examination of results of this type.

1.3. It is always possible to determine a polynomial of degree n which takes $n+1$ given values $\{y_s\}$, $s=1, 2, \dots, n+1$ at $n+1$ prescribed points $\{z_s\}$, $s=1, 2, \dots, n+1$ and there exists only one such polynomial of degree n . The analogue of this result for integral functions is that given any sequence $\{z_n\}$ of points and an associated sequence $\{y_n\}$ of values, it is always possible to find an integral function $F(z)$ such that $F(z_n) = y_n$ (II, § 2). But the solution is no longer unique. It is possible to find an infinity of integral functions satisfying the required conditions. Even if it be required that the solution in question should be of appropriate order and type, it is not possible to state in general whether such solutions exist and even if such do exist whether the solution is unique. But if suitable restrictions are placed on $\{z_n\}$ and $\{y_n\}$ it is possible to obtain significant results in this direction. It is this problem that forms the subject-matter of this thesis.

2. The first significant result in this connection was suggested by Polya and proved by Tachakaloff and others (theorem A, paper C).

* Multiple roots of $F(z) - a$ are to be counted according to their multiplicity.

It states that an integral function of order one and minimal type bounded at the points $z = \pm n$, $n = 0, 1, 2, \dots$ reduces to a constant. This result was, in fact, the starting-point of the investigations the results of which are embodied in this thesis. A similar result has been proved by Polya for functions of order two bounded at the lattice-points (paper D).

2.1. As a first generalisation of Polya's result, I have considered the question of replacing the points $[\pm n]$ by a general sequence $[\pm \lambda_n]$ $0 < \lambda_1 < \lambda_2 < \dots < \lambda_n \rightarrow \infty$. I prove that when $[\pm \lambda_n]$ is of order one and finite I.D. (11, § 1.1) and $l(\sigma) > 0$, $\sigma(z)$ being the associated canonical product, Polya's result holds when $[\pm n]$ are replaced by $[\pm \lambda_n]$. Next, I have considered the question whether similar results hold for functions of any finite order ρ . The answer is found in theorem 3 (C). It is found necessary to consider points lying on $2q$ lines inclined at equal angles π/q , q being an integer given by the inequality $\rho \leq q < \rho + 1$. In fact, if $[\lambda_n]$ is of order ρ and $[z_n]$ are the zeroes of

$$\sigma(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z^{2q}}{\lambda_n^{2q}} \right)$$

where $l(\sigma) > 0$ and $[\lambda_n^{2q}]$ is of finite I.D., then a function $F(z)$ of order ρ and minimal type such that $F(z_n) = O(1)$ must reduce to a constant. That, in general, it may not be possible to obtain a significant result of this type with less than $2q$ lines, is seen if we consider the function

$$G(z) = \int_0^a \phi(x) e^{-zx} dx,$$

where $a > 0$ is finite and $\phi(x)$ is L-integrable in $(0, a)$. Here $G(z)$ is an integral function of order one and *therefore of order two and minimal type*. It is bounded on the positive real axis and on the two halves of the imaginary axis; that is, on three lines. Therefore for functions of order two, less than 4 ($= 2q$) lines will not give a result of the type required.

2.2. It is well known that an integral function of order less than $1/2$ exhibits many peculiar properties and behaves more closely like polynomials than other integral functions. I have examined whether these functions exhibit any peculiarities in connection with the properties discussed here. The results are contained in theorems 1 and 2 of (C). I show that if $[z_n]$ is any sequence of order ρ , $0 < \rho < \frac{1}{2}$ and finite I.D. while $k(\sigma) < 0$, then an integral function of order ρ and minimal type reduces to a constant if $F(z_n) = O(1)$. In particular, if $[\lambda_n]$ is a measurable sequence of order ρ and density

$D > 0$ for which $\delta(\rho) = 0$ then the above result holds for any sequence $[z_n]$ for which $|z_n| = \lambda_n$. Again when we actually consider the measurable sequence $[\lambda_n]$ having the above properties, a function $F(z)$ of order ρ and type less than $\pi D \cot \pi \rho$ bounded at the points $z = \lambda_n$ must reduce to a constant, so that, in this case, we get a definite lower bound for the type of functions bounded at the points $[\lambda_n]$. These results along with those of the previous paragraph constitute the main advance on the initial result suggested by Polya.

2.3. It is a characteristic feature of the results mentioned above that the exponent of convergence of the sequence where the function in question is supposed to be bounded, is not by itself sufficient to lead to significant results. We have to suppose in addition that the I.D. is finite, and the density is not too small. The I.D. is a rough measure of the rate of increase of the sequence $[|1/\sigma'(z_n)|]$ which governs the relative distribution of the points of the sequence. The reason why some such additional assumption is necessary will become clear if we remember that, whereas the points at which a function takes a given value form an isolated set, the set of all points where $F(z)$ assumes one or other of the values of the bounded set $[y_n]$ is, in general, not isolated. Let $D_n = [z_{n1}, z_{n2}, \dots]$ be the set of points where $F(z) = y_n$. What we attempt to do in the problems discussed here is to characterize $F(z)$ by the nature of the sequence $[z_n]$ where z_n is 'picked out' from D_n . It is evident that an arbitrary set $[z_n]$, $|z_n| \rightarrow \infty$, picked out from $D_1, D_2, \dots, D_n, \dots$ will not in general characterise $F(z)$. The point z_n chosen from D_n must be representative not only of D_n but also of the relative situation of D_n with respect to the remaining sets $D_1, D_2, \dots$. This proper 'picking out' of the representative of each D_n may be regarded as giving rise to the necessity for some hypothesis on the I.D. or the index of condensation.

2.4. Again, besides the above properties of the sequence $[z_n]$, it will be seen from the proof of the results mentioned above that the maximum modulus curves of the canonical product play an important part in the determination of the properties of functions bounded at the points $[z_n]$. When the order is less than half this is not effectively seen since there are circles of arbitrarily large radii over which the minimum modulus is as large as the maximum modulus. But for $\rho \geq \frac{1}{2}$ this is no longer true. That is why we have had to assume that the points $[z_n]$ are distributed on one or more lines so that the maximum modulus curve is effectively determined.

2.5. Besides the above results, I prove in paper (D) that a function of order two and type less than $\frac{1}{2}\pi$ bounded at the lattice-points must be a constant. In paper (E) I discuss a slightly different problem. Instead of postulating the finiteness of the I.D. of $[z_n]$ I connect the value $[y.]$ with $[\sigma'(z_n)]$ which measures the I.D. Here also I obtain three results (Theorems 1-3 of paper E). Another result discussed in (E) is closely allied to a theorem due to Faber that for a function of order one and minimal type $[F(n)] > e^{-\alpha n}$ for any $\alpha > 0$ and for almost all positive integers. I show that if $[\lambda_n]$ is a positive increasing sequence tending to infinity for which $\lim_{r \rightarrow \infty} \frac{n(r)}{r} > 0$ and for which the index of condensation is zero, a function $F(z)$ of order one and minimal type cannot satisfy the relation $F(\lambda_n) = O(e^{-\alpha \lambda_n})$, $\alpha > 0$ unless $F(z) \equiv 0$. If in addition $[\lambda_n]$ is measurable and has density D while $F(\lambda_n) = O(e^{-\alpha \lambda_n})$ however large α be, then $k(F) > \pi D$ unless $F(z) \equiv 0$.

3. Finally it may be noted that all the results obtained here have, besides their explicit content, a bearing on uniqueness theorems. For instance consider theorem 2, paper (C). Let $[\lambda_n]$ be a measurable sequence satisfying the conditions of the theorem. Then any function of order ρ bounded at $[\lambda_n]$ must have its type not less than $\pi D \cot \pi \rho$ unless it is a constant which is a trivial case. On the other hand, suppose that there are two functions $F(z)$ and $\phi(z)$ of order ρ both of whose types are less than $\pi D \cot \pi \rho$. Then if the difference of their values at $z = \lambda_n$ does not numerically exceed a finite number, the two functions differ only by a constant. That is, there are not essentially two distinct functions with the required properties. All the other results also can be interpreted in a similar manner.

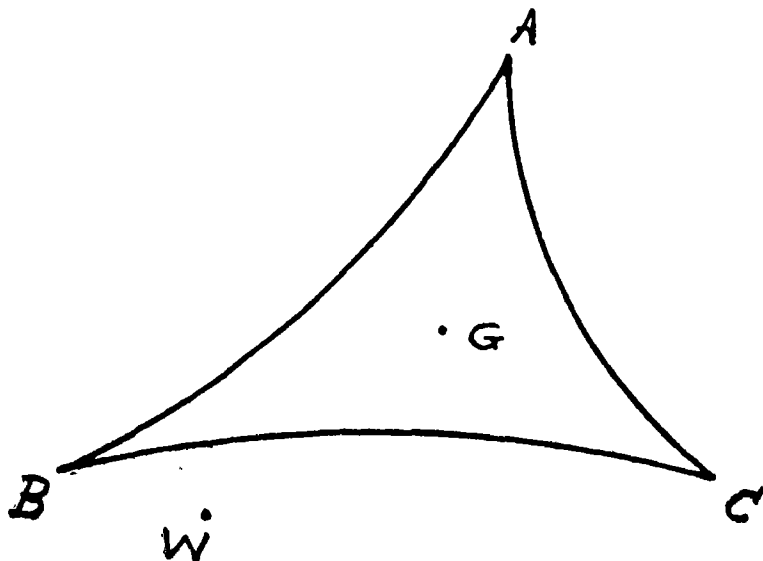
THE META-DELTOID

BY

N. DURAIRAJAN

1. *Introduction*

A curve of order four and class three which has the line at infinity as a bitangent is a biparabola. [R. 1]. It has three cusps and one bitangent. If all the cusps are real, then the points of contact of the bitangent are imaginary. If one cusp is real and two are imaginary, then the points of contact of the bitangent are real. [R. 2.]. The first positive pedal of the biparabola with respect to its unique focus is a conic. If the conic be an ellipse, the biparabola has three real cusps; if it is a hyperbola, one cusp is real and the other two are imaginary; if the hyperbola is rectangular, then the biparabola is rectangular or harmonic [R. 1]. The Deltoid or the three cusped hypocycloid is a limiting case and may be said to correspond to the circle.



THE META-DELTOID
(The position of W is approximate.)

The Deltoid touches the line at infinity at the circular points. If the deltoid is orthogonally projected, we get a biparabola, which

has three real cusps and in which the cuspidal tangents concur at the centroid of the triangle of cusps. I propose to call this curve, as the *meta-deltoid* and to discuss some of its properties. A tracing of the curve is attached.

2. The equation of the Curve

If the cuspidal triangle is taken as the triangle of reference, the general equation of a tricuspidal quartic is [R. 2]

$$(lx)^{-1/2} + (my)^{-1/2} + (nz)^{-1/2} = 0$$

when the cuspidal tangents concur at the centroid, the equation becomes

$$x^{-1/2} + y^{-1/2} + z^{-1/2} = 0 \quad \dots (1)$$

$$\text{or} \quad y^2 z^2 + z^2 x^2 + x^2 y^2 - 2xyz(x+y+z) = 0 \quad \dots (2)$$

The areal tangential equation is

$$l^{-1/3} + m^{-1/3} + n^{-1/3} = 0 \quad \dots (3)$$

$$\text{or} \quad (l+m+n)^3 - 27lmn = 0 \quad \dots (4)$$

$$\text{or} \quad p_6^3 = p_1 p_2 p_3 \quad \dots (5)$$

where p_6, p_1, p_2, p_3 are the lengths of the perpendiculars on a tangent from the centroid and the three vertices.

The curve is the isotomic transformation of the Steiner in-conic, whose equation is

$$x^{1/2} + y^{1/2} + z^{1/2} = 0 \quad \dots (6)$$

and which cuts the line at infinity at the points of contact of (1) with its bitangent which is $x+y+z=0$.

3. The position of the focus

The foci of (4) are given by substituting $l = z - \alpha$, $m = z - \beta$, $n = z - \gamma$, where z, α, β, γ are the complex numbers representing the focus and the three vertices. [R. 3.]

$$\therefore (3z - \alpha - \beta - \gamma)^3 = 27(z - \alpha)(z - \beta)(z - \gamma) \quad \dots (7)$$

This reduces to a linear equation i.e. there is one unique focus, which is given by

$$z = \frac{(\alpha + \beta + \gamma)^3 - 27\alpha\beta\gamma}{9(\alpha^3 + \beta^3 + \gamma^3 - \beta\gamma - \alpha\beta - \gamma\alpha)} \quad \dots (8)$$

Remembering that the complex co-ordinate of the centroid and the two Hessian points are

$$\begin{aligned} g &= \frac{1}{3}(\alpha + \beta + \gamma) \\ h_1 &= -\left(\frac{\beta\gamma + \omega\gamma\alpha + \omega^2\alpha\beta}{\alpha + \omega\beta + \omega^2\gamma}\right) \\ h_2 &= -\left(\frac{\beta\gamma + \omega^2\gamma\alpha + \omega\alpha\beta}{\alpha + \omega^2\beta + \omega\gamma}\right) \end{aligned} \quad \dots (9)$$

where ω is a complex cube root of unity, we get

$$z = \frac{1}{3}(g + h_1 + h_2) \quad \dots (10)$$

Hence the focus, W, is the centroid of the triangle whose corners are the centroid and the Hessian points of the triangle of cusps. We can write (7) in the form

$$(z - g)^3 - (z - \alpha)(z - \beta)(z - \gamma) \quad \dots (11)$$

Equating moduli and arguments, we get the following properties

$$WG^3 = WA \cdot WB \cdot WC \quad \dots (12)$$

$$\angle AWG + \angle BWG + \angle CWG = 0 \pmod{2\pi} \quad \dots (13)$$

Taking the vertices as origin in succession, we get

$$AG^3 = 3\sigma^2 \cdot AW$$

$$BG^3 = 3\sigma^2 \cdot BW$$

$$CG^3 = 3\sigma^2 \cdot CW$$

$$AG \cdot BG \cdot CG = 3\sigma^2 \cdot GW \quad \dots (14)$$

where $\sigma = GS$, S being a focus of the Steiner in-conic. The vector GS is given by

$$GS = g + \frac{1}{3}\sqrt{\alpha^2 + \beta^2 + \gamma^2 - \beta\gamma - \gamma\alpha - \alpha\beta}.$$

The foci of the Steiner in-conic are given by

$$\frac{1}{z - \alpha} + \frac{1}{z - \beta} + \frac{1}{z - \gamma} = 0 \quad [\text{R. 3}].$$

4. The pedal and orthoptic loci

The tangential equation of the biparabola referred to the focus as origin is [R. 1] in rectangular cartesian

$$(al + bm)(l^2 + m^2) + n(l^2 + \mu m^2) = 0 \quad \dots (15)$$

When $\mu > 0$, the curve is a meta-deltoid. The first positive pedal with respect to the focus is [R. 1.]

$$C_p = x^2 + \mu y^2 - ax - by = 0 \quad \dots (16)$$

The orthoptic locus is the conic

$$C_0 = (x - a)[x(1 + \mu) - 2a] + (y\mu - b)[y(1 + \mu) - 2b] = 0 \quad \dots (17)$$

The two conics are similar.

5. *Miscellaneous Properties*

(i) If PQR be a maximum inscribed triangle in (16), it is well known that the circles of curvature at P, Q, R intersect the conic again at a point W, and that W, P, Q, R are concyclic. The negative pedal of the ellipse with respect to W is a meta-deltoid whose cuspidal tangents are the lines through P, Q, R, perpendicular to WP, WQ, WR respectively. The point of concurrence of the cuspidal tangents is W', the point diametrically opposite to W in the circle WPQR.

(ii) If a circle through W cuts the ellipse in X, Y, Z, then the lines LX, LY, LZ perpendicular to WX, WY, WZ respectively are tangents to the meta-deltoid, intersecting at L on the circle WXYZ.

(iii) Consider the circle of curvature at W to the ellipse. The normal to the ellipse at W touches the meta-deltoid at a point, which is the centre of curvature of the ellipse at W.

(iv) The ellipse is tritangent to the meta-deltoid.

(v) The maximum and minimum tangent-chords of the meta-deltoid are parallel to the axes of the steiner circumconic of the triangle ABC and their lengths are two-thirds of the lengths of these axes. This is easily derived from the property that tangent-chords of a deltoid are constant in length and that their orthogonal projections are maximum and minimum for lines parallel and perpendicular to the line of the intersection of the two planes.

(vi) The focus W of the meta-deltoid lies outside the curve, as two of the three tangents from W to the curve are lines joining it to the circular points at infinity.

References

- R. 1. *Jour. Ind. Math. Soc.* Vol. 17 (1927) P. 77.
- R. 2. BASSET *Cubic and Quartic Curves.* P. 132. 1901 Edition.
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NOTES AND DISCUSSIONS

A Note On The Nine Points Circle

THEOREM — *If Σ be a conic inscribed in a triangle and Σ' the confocal to Σ inpolar to the circumcircle of the triangle the mutual orthoptic circle of Σ and Σ' cuts orthogonally the nine points circle of the triangle*

It is well known that in any range (4 line system) of conics

$$\Sigma_{\lambda} \quad \Sigma + \lambda \Sigma = 0$$

there is in general, a unique member inpolar to a given conic locus. It follows from this that given any pencil (4 point system) of conics

$$S_{\mu} \quad S + \mu S' = 0$$

we can associate with every member S_{μ} of the pencil a unique member Σ_{λ} of the range such that Σ_{λ} is inpolar to S_{μ} . Thus the parameter λ of the range is in homographic correspondence with the parameter μ of the pencil.

Now, the polar circle S of the triangle, the circumcircle, S' and the nine points circle S'' belong to a coaxal system. The radical axis of the system (together with the line at infinity) is a circle S'' whose centre is the point at infinity on the line of centres. Thus, the orthocentre s , the circumcentre s' , the nine points centre s'' and the point at infinity on this line, say s''' , are in homographic correspondence with the circles S, S', S, S' . But, $(ss', s's'') = -1$. Hence S' is the harmonic conjugate of S'' w.r.t. S and S .

Let Σ' be the harmonic conjugate of the absolute conic Σ (consisting of the pair of circular points at infinity of the Euclidean plane) w.r.t. Σ and Σ' . Now, the F-conic of any two conics X, Y passes through the points of intersection of pairs of conics like $X + \alpha Y = 0, X - \alpha Y = 0$ in the range determined by X and Y . Thus, the F-conic of Σ'', Σ''' viz., the director circle of Σ passes through the common points of Σ, Σ' and is, therefore, the mutual orthoptic circle of Σ, Σ' .

Now, the conics Σ of the confocal system are in homographic correspondence with the outpolar members of the coaxal system of circles S . Thus, S, S', S'' are outpolar to $\Sigma, \Sigma', \Sigma'''$ respectively. Hence S'' the harmonic conjugate of S''' w.r.t. S and S' is outpolar to the corresponding member Σ'' in the confocal system. Therefore,

by Gaskin's Theorem the nine points circle S'' cuts orthogonally the director circle of Σ'' which is the mutual orthoptic circle of Σ, Σ' .

In particular, if Σ' consists of a pair of points they are a pair of foci of Σ and are, therefore, isogonal conjugates w.r.t. the triangle. Further, the mutual orthoptic circle of Σ, Σ' in this case is evidently the pedal circle of the pair of isogonal conjugates. We have thus the following extension of Feuerbach's Theorem by Shusakn Ogino. (*vide Tohoku. Math. Jour. Vol. 43, p. 147.*)

If a pair of isogonally conjugate points w.r.t. a triangle are also conjugate w.r.t. the circumcircle, their pedal circle cuts orthogonally the nine points circle of the triangle.

Annamalainagar, }
27—11—38. }

K. RANGASWAMI.

The Twisted Straight Line

Perhaps the best comment on Narasinga Rao's note (*Mathematics Student*, vol. 6, p. 74) is to point out the simplest application of the idea of the twisted line. The two assertions:

The osculating plane at any point of an asymptotic line on a surface is the tangent plane to the surface.

The osculating plane at any point of a geodesic on a surface contains the normal to the surface;

can both be made without the reservation which careful writers have added that the line in question must not be a straight line. In the case of a straight line the assertions define the plane which is to serve as osculating plane according to the part which the line is considered as playing; there is no contradiction in the fact that the plane which corresponds to the one role is at right angles to the plane which corresponds to the other role.

As an example of the use of the first theorem, if the parameter of distribution of a generator of a ruled surface is p , the angle ν between the normal at the central point of the generator and the normal at the point whose distance from the central point is r is given by $\tan \nu = r'p$. Regarded as an asymptotic line the generator therefore has torsion $p'(r^2 + p^2)$, and it follows without further investigation that the curved asymptotic line has torsion $-p.(r^2 + p^2)$ and that the Gaussian curvature of the surface is $-p^2/(r^2 + p^2)^3$.

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A Note on the Principal Axes of a Conic

1. The necessary and sufficient condition that the straight line whose co-ordinates are (l, m, n) should be a diameter of the conic whose areal-tangential equation is

$$Al^2 + Bm^2 + Cn^2 + 2 Fmn + 2 Gnl + 2 Hlm = 0,$$

is that its pole, i.e., the point

$$(Al + Hm + Gn, Hl + Bm + Fn, Gl + Fm + Cn),$$

should lie on the line at infinity.

This condition is

$$\sum (A + H + G)l = 0 \quad \dots (1)$$

If, further, the line is a principal axis of the conic it is perpendicular to its conjugate diameter for which the condition is :

$$\sum (Al + Hm + Gn)(m - n) \cot \alpha = 0 \quad \dots (2)$$

where α, β, γ are the angles of the triangle of reference.

2. If the line (l, m, n) is a principal axis of the conic

$$Al^2 + Bm^2 + Cn^2 = 0,$$

then from equations (1) and (2), above, we see that

$$Al : Bm : Cn = M - N : N - L : L - M, \quad \dots (3)$$

where

$$\begin{aligned} L &= (n - l) \cot \beta - (l - m) \cot \gamma, \\ M &= (l - m) \cot \gamma - (m - n) \cot \alpha, \\ N &= (m - n) \cot \alpha - (n - l) \cot \beta. \end{aligned}$$

For a system of conics touching four straight lines, taking the common diagonal triangle for the triangle of reference, the four straight lines have co-ordinates of the form $(l_0, \pm ml_0, \pm nl_0)$; and the tangential equation of any conic of the system is

$$Al^2 + Bm^2 + Cn^2 = 0,$$

where

$$Al_0^2 + Bm_0^2 + Cn_0^2 = 0 \quad \dots (4)$$

Hence, if the line (l, m, n) be a principal axis of a conic of the system, then from (3) and (4) we see that

$$\sum \{l_0^2 (M - N)/l\} = 0$$

i.e.

$$\sum l_0^2 mn(M - N) = 0 \quad \dots (5),$$

which is therefore the tangential equation of the envelope of the axes of conics of the system. It is of class three.

In the particular case, when one conic of the system is a circle, its tangential equation is

$$\sum l^2 \tan \alpha = 0.$$

Hence from (4)

$$\sum_0 l^2 \tan \alpha = 0 \quad \dots (6)$$

Using (6), equation (5) becomes

$$\left[\sum l \tan \alpha \right] \left[\sum l_0^2 mn \tan \alpha \right] = 0 \quad \dots (7)$$

Hence, the envelope breaks up into

(i) the point $(\tan \alpha, \tan \beta, \tan \gamma)$, which is the centre of the circle and the orthocentre of the triangle of reference; and

(ii) the parabola

$$\sum_0 l^2 mn \tan \alpha = 0,$$

which touches the sides of the triangle of reference.

For a system of conics through four points, taking the common self-conjugate triangle for the triangle of reference, the four points have co-ordinates of the form $(x_0, \pm y_0, \pm z_0)$; and the tangential equation of any conic of the system is

$$Al^2 + Bm^2 + Cn^2 = 0,$$

where
$$x_0^2/A + y_0^2/B + z_0^2/C = 0 \quad \dots (8)$$

Hence from (3) and (8), the tangential equation of the envelope of the principal axes of conics of the system is

$$\sum x_0^2 l/(M - N) = 0,$$

i.e.
$$\sum x_0^2 l(L - M)(L - N) = 0, \quad \dots (9)$$

which is of class three.

In the particular case, when one conic of the system is a circle, we have, as in (6),

$$\sum_0 x_0^2 \cot \alpha = 0 \quad \dots (10)$$

Using (10), equation (9) becomes

$$\left[\sum l \cot \beta \cot \gamma \right] \left[\sum_0 x_0^2 (l - m)(l - n) \right] = 0.$$

Hence, the envelope breaks up into

- (i) the point $(\tan \alpha, \tan \beta, \tan \gamma)$, the centre of the circle; and
- (ii) two points at infinity represented by the equation

$$\sum a_0^2 (l-m)(l-n) = 0.$$

From (ii) we see that if one conic of a four point system is a circle, then the principal axes of all conics of the system are parallel—a result which follows from the well-known theorem “A pair of common chords of a circle and a conic are equally inclined to the principal axes of the conic.”

PROF. Watson has obviously made a slip on p. 263, Vol. XXII of the *Mathematical Gazette*, where he states that (ii) is a parabola.

3. From the above result we see that if the line $lx + my + nz = 0$ is a principal axis of the conic

$$S \equiv (a, b, c, f, g, h)(x, y, z)^2 = 0,$$

then, one conic of the system

$$kS - (lx + my + nz)^2 = 0$$

must be a circle.

Therefore

$$\frac{k\rho - (m-n)^2}{a_0^2} = \frac{kq - (n-l)^2}{b_0^2} = \frac{kr - (l-m)^2}{c_0^2},$$

where $\rho = b + c - 2f$, $q = c + a - 2g$, $r = a + b - 2h$, and a_0, b_0, c_0 , are sides of the triangle of reference.

Eliminating k from the above equations, the co-ordinates (l, m, n) of the line satisfy the equation

$$\begin{vmatrix} (m-n)^2 & \rho & a_0^2 \\ (n-l)^2 & q & b_0^2 \\ (l-m)^2 & r & c_0^2 \end{vmatrix} = 0,$$

which is the tangential equation of the points at infinity on the principal axes.

4. The equation of the principal axes of the conic

$$(a, b, c, f, g, h)(x, y, z)^2 = 0,$$

can be obtained in the form

$$\begin{vmatrix} (Y-Z)^2 & p & a^2 \\ (Z-X)^2 & q & b^2 \\ (X-Z)^2 & r & c^2 \end{vmatrix} = 0,$$

where

$$X = ax + hy + qz,$$

$$Y = hx + by + fz,$$

$$Z = qx + fy + cz.$$

For, if x, y, z , be any point on a principal axis, then one conic of the system

$$\lambda S - (xX_1 + yY_1 + zZ_1)^2 = 0,$$

must be a circle.

The equation of the axes can also be written in the form

$$\begin{vmatrix} (X-Y)(X-Z) & q+r-p & \cot \alpha \\ (Y-Z)(Y-X) & r+p-q & \cot \beta \\ (Z-X)(Z-Y) & p+q-r & \cot \gamma \end{vmatrix} = 0.$$

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GLEANINGS

In mathematics we use many easy words: "group," "family," "ring," "simple curve." But these ordinary words of English are given a very peculiar and difficult technical meaning. In fact here is a booby prize definition for mathematics: *Mathematics is the science which uses easy words for hard ideas.*

EDWARD KASNER: *Essay on New Names in Mathematics.*

Two Fundamental Inequalities

I. If x, p, q be all positive, then

$$\frac{x^p - 1}{p} \geq \frac{x^q - 1}{q} \quad \left(p \geq q \right).$$

II. If $x > 0$ and $x \neq 1$, then

$$(1) \quad m x^{m-1} (x-1) > x^m - 1 > m (x-1), \quad [m < 0, \text{ or } m > 1],$$

$$\text{and } (2) \quad m x^{m-1} (x-1) < x^m - 1 < m (x-1), \quad [0 < m < 1].$$

The following proofs of these inequalities are based on the calculus and are, on that account, less elementary and less fundamental. It will be seen that the proofs assume the formula $dx^n/dx = nx^{n-1}$ when n is irrational, which is not quite an elementary result. But the proofs now become quite easy.

Proof of I. If $x > 1$, then $x^{p-q} \geq 1$ for $p \geq q$. This inequality is equivalent to $x^{p-1} \geq x^{q-1}$ for $p \geq q$. Integrating from 1 to x we have

$$\frac{x^p - 1}{p} \geq \frac{x^q - 1}{q} \quad \text{for } p \geq q \text{ and } x > 1$$

If $0 < x < 1$, then $x^{p-q} \leq 1$ for $p \geq q$, and $x^{p-1} \leq x^{q-1}$. Integrating from x to 1 since $x < 1$, we have

$$\frac{1 + x^p}{p} \leq \frac{1 + x^q}{q}$$

$$\text{Hence} \quad \frac{x^p - 1}{p} \geq \frac{x^q - 1}{q}, \quad \text{for } p \geq q \text{ and } x < 1$$

Proof of II. (1) Suppose, in the first place, $x > 1$. Then, for all real values of m we have $x^{m-1} > x^{m-2}$. Integrating from 1 to x we have

$$\frac{x^m - 1}{m} > \frac{x^{m-1} - 1}{m-1}.$$

If $m > 1$ or $m < 0$ this gives on multiplication by $m(m-1)$ which is positive

$$(m-1)(x^m - 1) > m(x^{m-1} - 1)$$

that is,

$$m x^{m-1} (x-1) > x^m - 1;$$

The first result of II (1) is therefore proved for $x > 1$.

To prove the other part of II (1) we note that for $x > 1$, $m > 1$, $x^{m-1} > 1$, which, on integration from 1 to x gives $x^m - 1 > m(x-1)$; and that if $x > 1$, $m < 0$, $x^{m-1} < 1$, which gives on integration from 1 to x ,

$$\frac{x^m - 1}{m} < x - 1,$$

or $x^m - 1 > m(x - 1)$, since $m < 0$.

Hence in either case $x^m - 1 > m(x - 1)$, and II (1) is proved completely for $x > 1$. The substitution $y = 1/x$, $x > 1$, will show that the same inequality is true for $0 < y < 1$. The inequality II (2) can be proved similarly.

Lahore. }

VIDYA PARKASH MALHOTRA.

SOLUTIONS TO QUESTIONS

Question 1699

(A. RANGANATHA RAO):—If O is a given point in the plane of a conic S, prove that :

(i) there exist four points A, B, C, D on S such that the directrix of the unique parabola having third order contact with S at each of these points, passes through O.

(ii) A, B, C, D lie on a circle centre O.

(iii) the circle ABCD is out-polar to S.

(iv) ABCD is the unique cyclic tetrad on S with centre O, apolar to the feet of the normals from O to S.

Solution by K. Rangaswami.

Let σ be a parabola having third order contact with S at a point P and whose directrix passes through O. One pair of opposite vertices of the quadrilateral circumscribed to σ and S is the coincident pair of points PP. Hence the coaxal system determined by the director circle of S and the directrix of σ has P for a limit point.

Now, any circle with centre O cuts orthogonally the directrix of σ ; in particular, the unique circle Σ with centre O and out-polar to S also cuts orthogonally, by Gaskin's Theorem, the director circle of S. Thus, Σ belongs to the conjugate coaxal system and, therefore, passes through the limiting point P of the first system. Hence the four points A, B, C, D common to Σ and S are the possible positions for P.

Further if Γ be the Steiner parabola of O w. r. t S , then O lies on the directrix of Γ . Hence the circle Σ is out polar to Γ . But Σ is also out-polar to S . Therefore, the intersection tetrad of Σ and S is apolar to the contact tetrad of S and Γ i.e., the cyclic tetrad $ABCD$ is apolar to the pedal tetrad of O w. r. t S .

Also solved analytically by the Proposer.

Question 1743

(N. DORAIRAJAN).—If P be any point on the edge CD of a tetrahedron $ABCD$, show that

(i) the locus of the circumcentre of the triangle PAB is a plane nodal circular cubic;

(ii) the locus of the orthocentre of the triangle PAB is a skew cubic and that it becomes a circle if the lines AB, CD are at right angles.

Solution by C. N. Srinivasengar.

The first part is identical with Ex. 12. P. 203 in Hilton's *Plane Algebraic Curves*. The locus (i) is evidently a curve lying in the plane bisecting AB at right angles. If O be the mid-point of AB , and if the sphere on AB as diameter cuts CD in H and K , the circumcentre of each of the triangles ABH, ABK is O itself. Hence O is a node on the locus. Next consider any line OL perpendicular to AB . The plane OL, AB meets CD in one point P such that the circumcentre of $\triangle ABP$ lies on OL . Thus OL meets the locus twice at O (since O is a node) and once elsewhere. The locus is thus a plane cubic with a node at O .

To obtain the locus analytically, choose O as origin, and the points A, B as $(0, 0, \pm a)$. The circle with centre at $(x_1, y_1, 0)$ and passing through A and B is given by

$$x^2 + y^2 + z^2 - 2xx_1 - 2yy_1 - a^2 = 0; \quad xy_1 - x_1y = 0.$$

By finding the condition that this circle intersects CD given by $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$, we easily get the locus of $(x_1, y_1, 0)$ in the form $2(l\beta - m\alpha)(ly - mx)(x^2 + y^2) = (\beta x - \alpha y)^2 + 2A(\beta x - \alpha y)(ly - mx) + B(ly - mx)^2$, where $A = l\alpha + m\beta + n\gamma$, $B = \alpha^2 + \beta^2 + \gamma^2 - a^2$.

We thus get a nodal circular cubic.

(2) The orthocentre is the intersection of the plane ABP , the plane through A perpendicular to BP , and the plane through B

perpendicular to AP. As P varies, these three planes evidently form three pencils through AB, the altitude from A on plane BCD, and the altitude from B on plane ACD, such that there is a (1, 1) correspondence between the planes of any two pencils. Hence the locus of the orthocentre is a twisted cubic in general.

Choosing axes etc. as before, the locus is easily found to be

$$x = \{a^2 - (\gamma + nr)^2\}(\alpha + lr) \div d$$

$$y = \{a^2 - (\gamma + nr)^2\}(\beta + mr) \div d$$

$$z = \gamma + nr,$$

where r is the parameter, and $d = (\alpha + l)^2 + (\beta + m)^2$. If AB is perpendicular to CD, $n=0$, and the locus reduces to

$$z = \gamma; \quad x^2 + y^2 = (a^2 - \gamma^2)^2 u$$

The locus of the centroid of ABP is evidently a line parallel to CD. Since the nine-points centre is a point of trisection of the join of the centroid and the orthocentre, the locus of the nine points centre will in general be a twisted cubic.

Question 1746

(MISS S. PANKAJAM):—If x and y be any two given integers and r an unknown integer, find (a) the integers r

and

(b) the number of such values of r

for which

(i) $\binom{x}{r}$ is odd,

(ii) $\binom{x}{r} - \binom{y}{r}$ is odd;

(iii) $\binom{x}{r} + \binom{x}{r-2}\binom{y}{2} + \binom{x}{r-4}\binom{y}{4} + \dots$ is odd

[$\binom{x}{r}$ etc. are the usual binomial coefficients.]

Solution by the Proposer.

(i) We express x in the binary scale so that

$$x = 2^m + 2^n + 2^p + \dots \text{ to } \sigma(x) \text{ terms } (m > n > p > \dots > 0).$$

Now, since $(1+z)^2 \equiv (1+z^2) \pmod{2}$

therefore, $(1+z)^4 \equiv (1+z^2)^2 \pmod{2}$

And generally, $(1+z)^{2^t} \equiv 1 + z^{2^t} \pmod{2}$

... (I)

Hence,

$$\begin{aligned}
 (1+z)^x &\equiv (1+z)^{2^m} (1+z)^{2^n} \cdots (1+z)^{2^p} \cdots \text{ to } \sigma(i) \text{ terms} \\
 &= (1+z^{2^m})(1+z^{2^n})(1+z^{2^p}) \pmod{2} \\
 &= 1 + z^{2^m} + z^{2^n} + \cdots + \sum z^{2^m + 2^n} + \sum z^{2^m + 2^n + 2^p} \quad \dots (II)
 \end{aligned}$$

(a) It is obvious that the values of i for which $\binom{i}{r}$ is odd, are the powers of z which occur in II. These powers are numbers of the form $2^m + 2^n + \cdots$ i.e. the numbers which are obtained by replacing any number of units by zeros in the expression of i in the binary scale.

(b) The total number of numbers i (including zero) for which $\binom{i}{r}$ is odd, is evidently $2^{\sigma(i)}$, where $\sigma(r)$ is the number of units in the expression of r in the binary scale.

(ii) Let us express i , y , and r in the binary scale and let λ_{00} , λ_{11} , λ_{10} , λ_{01} denote the sets of places in which there are zeros of both i and y , units of both x and y , units of i against zeros of y , zeros of i against units of y respectively.

Then, from the form of the expressions in the binary scale and from (i), the following properties are evident

(1) For $\binom{i}{r}$, $\binom{y}{r}$ to be both odd, r has zeros at λ_{00} , λ_{10} , λ_{11} .

(2) For $\binom{i}{r}$ odd and $\binom{y}{r}$ even, r has zeros at λ_{00} , λ_{01} , and has at least one unit in the set of places λ_{10} ; and for $\binom{i}{r}$ even and $\binom{y}{r}$ odd, r has zeros at λ_{10} , λ_{00} , and one unit at least in the set of places λ_{01} .

(3) For $\binom{i}{r}$, $\binom{y}{r}$ to be both even, r must have units either at the places λ_{00} or at λ_{01} and λ_{10} .

We could now easily find (b). For, the number of values of r for which $\binom{i}{r} - \binom{y}{r}$ is odd = the number for which either $\binom{i}{r}$ or $\binom{y}{r}$ is odd, and the other even. Direct application of (2) gives the actual result. [(1) and (3) give the number of values of r for which $\binom{x}{r} - \binom{y}{r}$ is even] (a) cannot be solved in the general case. But

if x and y be given numerical values, x_1, y_1 we can find the actual values of r for which $\binom{x_1}{r} - \binom{y_1}{r}$ is odd.

(iii) I have been unable to solve this for any general x , and y .

Question 1750

(C. N. SRINIVASIENGAR):—Prove the following properties of scrolls:—

(1) The tangent lines of a scroll at points along a generator and at right angles to it generate a rectangular paraboloid, which is closely associated with the locus of normals at points along the generator;

(2) the line of striction of either system of generators of these paraboloids is a straight line;

(3) The osculating quadric along a generator of a scroll always touches the scroll "internally";

(4) The osculating quadric along any generator of a right helicoid is a rectangular paraboloid;

(5) If at points along a generator of a general scroll tangent lines are drawn to the orthogonal trajectories of the curved asymptotic lines, these lines generate a quartic scroll having the given generator as a triple line. The quartic scroll reduces to a cubic in special cases e.g. for a right conoid (excluding the right helicoid).

Solution by V. Rangachariar

Take the central point of the generator as the origin of rectangular co-ordinates, the generator as the x -axis and the normal at the central point as the z -axis. The tangent plane at $(u, 0, 0)$ is $\beta z = uy$ where β is the parameter of distribution. The equations of the tangent line at right angles to the generator are

$$\frac{x-u}{0} = \frac{y}{\beta} = \frac{z}{u}.$$

This line generates the rectangular paraboloid $xy = \beta z$, which is evidently closely associated with the locus of normals along the generator, viz. $xz = -\beta y$.

The line of striction of the generators

$$\frac{x-u}{0} = \frac{y}{\beta} = \frac{z}{u}.$$

of the paraboloid $xy = \beta z$ is clearly the x -axis. The other system of generators is given by

$$\frac{x}{\beta} = \frac{y - v}{0} = \frac{z}{v}$$

whose line of striction is the y -axis i.e. the tangent line through the central point at right angles to the generator. Similarly for the other paraboloid the two lines of striction are the x -axis and the z -axis.

All the generators of the right helicoid are parallel to the same plane, and hence the osculating quadric which is the limit of the quadric through three consecutive generators is a paraboloid. The right helicoid being a minimal surface, the given generator is at right angles to all the other inflexional tangents to the surface at points along the generator. Hence the osculating quadric is a rectangular paraboloid. It may be noted that since the right helicoid is the only real ruled minimal surface, it is the only ruled surface for which the osculating quadric along an arbitrary generator is a rectangular paraboloid.

Taking the line of striction of the generators of a scroll as the directrix, the equation of the scroll is $\mathbf{R} = \mathbf{r} + u\mathbf{d}$, with the usual vector notation. Using Weatherburn *Differential Geometry* Vol. I, Art. 73 and Art. 24, Ex. 1, the curved asymptotic line cuts the generator at an angle ψ given by the equation

$$\tan \psi = \frac{H}{Au^2 + Bu + C - \cos \theta}$$

where

$2AD = [\mathbf{d}, \mathbf{d}', \mathbf{d}'']$, $2BD = \kappa [\mathbf{d}, \mathbf{d}', \bar{\mathbf{n}}] + [\mathbf{d}, \mathbf{t}, \mathbf{d}'']$, $2CD = \kappa [\mathbf{d}, \mathbf{t}, \mathbf{n}]$ κ and $\bar{\mathbf{n}}$ being the curvature of the line of striction and its unit principal normal, the rest of the notation being as in Weatherburn. We know further that $\beta = \frac{D}{a^2} = \pm \frac{\sin \theta}{a}$, and hence

$$H^2 = a^2 u^2 + \sin^2 \theta = a^2(u^2 + \beta^2).$$

Taking now the co-ordinate axes as before, the equations of the tangent line perpendicular to the curved asymptotic line are

$$\frac{x - u}{\tan \psi \sqrt{(u^2 + \beta^2)}} = \frac{y}{\beta} = \frac{z}{u}.$$

This generates the quartic scroll

$$(xy - \beta z)[A\beta^2 z^3 + B\beta yz + (C - \cos \theta)y^3] = a\beta y^2(y^2 + z^2).$$

For a right conoid, the double line is the line of striction and is at

right angles to all the generators. Hence, $\theta = \pi/2$, $\kappa = 0$, and therefore $C = 0$, $A = 0$. The quartic now degenerates into the cubic scroll

$$Bz(xy - \beta z) = \alpha y(y^2 + z^2)$$

and the plane $y = 0$.

Solution by the Proposer

Consider a given generator $\iota = \text{constant}$ on the ruled surface

$$x = p(v) + u l(\iota), \quad y = q(v) + u m(\iota), \quad z = r(\iota) + u n(\iota).$$

At any point along this generator the equations of the tangent line to the surface, in the direction determined by $du/dv = \alpha + \beta u + \gamma u^2$, where α, β, γ are functions of ι i.e. constants for the generator, are

$$\frac{x - p - ul}{p' + ul' + l(\alpha + \beta u + \gamma u^2)} = \frac{y - q - um}{q' + um' + m(\alpha + \beta u + \gamma u^2)} = \frac{z - r - un}{r' + un' + n(\alpha + \beta u + \gamma u^2)}.$$

We can eliminate u between these equations by putting each ratio equal to 1, ρ , and solving for ρ, u , and $\alpha + \beta u + \gamma u^2 + \rho n$. We will obtain a quadric surface whose equation is of the form

$$R(K + \gamma R) + Q(\beta R + \alpha Q - P) = 0 \quad \dots (a)$$

where P, Q, R are linear functions of $x - p, y - q, z - r$ and where k is a constant which does not vanish for a non-torsal generator.

Equation (a) determines an ∞^3 of quadrics (as α, β, γ vary) all of which touch the given scroll throughout the length of the given generator. A particular case is when α, β, γ correspond to the second inflexional tangent at the point (u, v) . The quadric obtained in this case is the osculating quadric whose distinguishing property is that the contact with the scroll along the generator is everywhere of the second order.

Consider now the tangential direction at right angles to the generator $v = \text{constant}$. For this direction $du/dv = -F'/E$, so that $\beta = 0, \gamma = 0$. The quadric (a) now becomes a paraboloid. If the generators of this paraboloid at right angles to the given line be rotated about it through a right angle, they become the normals to the scroll along the given generator.

One of the lines of striction of the paraboloid is evidently the given generator. I have elsewhere* given the result : if the line of striction of one system of generators of a quadric be a straight line, the same is true about the line of striction of the other system; further the surface is a rectangular paraboloid i.e. its section at infinity consists of two lines conjugate to the circle at infinity.

* Bull. Calcutta Math. Soc. 28 (1936), 51.

Consider next the orthogonal trajectories of the curved asymptotic lines. If α, β, γ take values corresponding to the curved asymptotic line at (u, v) , then for the orthogonal trajectory we have

$$\frac{du}{dv} = - \frac{F(\alpha + \beta u + \gamma u^2) + G}{E(\alpha + \beta u + \gamma u^2) + F} = \frac{a_1 + b_1 u + c_1 u^2}{a_2 + b_2 u + c_2 u^2}$$

where a_1, a_2 etc. are functions of v . Eliminating u from the equations of the corresponding tangent by adopting a method similar to that used for getting equation (a), we get

$$Q^2(a_1 Q^2 + b_1 QR + c_1 R^2) = (PQ - KR)(a_2 Q^2 + b_2 QR + c_2 R^2).$$

This leads to the property (5) in the problem. The quartic scroll reduces to a cubic when $\gamma = 0$ i.e. when $c_1 = c_2 = 0$, in particular for the general right conoid.

To prove (3), choose $x - y = 0$ as the generator, so that the equation of the scroll is

$$2(a_0 x + a_1 y) + (ax^2 + by^2 + 2fyz + 2gzx + 2hxy) + \text{higher terms} = 0.$$

At any point $(0, 0, \lambda)$ the inflexional tangents are given by $x(a_0 + g\lambda) + y(a_1 + f\lambda) = 0$; $ax^2 + by^2 + 2xy + 2fyz + 2gzx + 2hxy = 0$. The osculating quadric is therefore

$$2(a_0 x + a_1 y) + (ax^2 + by^2 + 2fyz + 2gzx + 2hxy) = 0.$$

For given plane parallel to the tangent plane and close to it, the indicatrices of the scroll and the quadric are identical. Hence the surfaces touch internally* (4) follows from the same property as was used to prove (2). The right helicoid being a minimal surface, a given generator is a line of striction of the osculating quadric along it. Hence the quadric is a rectangular paraboloid.

* For the meaning of, and conditions for *internal* and *external* contact between two surfaces, see my article in *Math. Student* 4, 68-71, also A. Narasinga Rao, *Ibid.* 71-77.

REVIEWS

R. LAMBERT: *Structure Generale des Nomogrammes et des Systemes Nomographiques* 1937 pp. 64, 15 francs (Act. Sc. et Indust. No. 493, Hermann et Cie)

P. LUCKEY: "*Nomographie*": *Practical Guide for the Design of Graphical Calculators*, 3rd edn., 1937. pp. 108. (Math.-Phys. Bibl. I, Bd. 59/60. Paper covers, R. M. 2. 40)

The titles of these books give an indication of their complementary nature. In the first, though it is well illustrated, the only examples of a practical nature are two which concern finance and are taken from d'Ocagne's works. In the second, besides sketches for the theory, about 25 problems are solved by fully worked nomograms, several of them in more than one way, and these, save one for compound interest, are concerned with physics and engineering.

In a preface to the first d'Ocagne points out that it is a study in the morphology of the nomogram, and says that the most original part of the work is the investigation of nomographic systems. "Resolving these necessities the use of adjustments. By a discriminating and systematic treatment the author succeeds in simplifying and facilitating these adjustments

This work should be followed by a number of others of the same type, by different writers, to expound the various processes, both mechanical and graphical, by which numerical calculations can be simplified and expedited." The fundamental ideas used are contact and degrees of freedom, and for a final classification and enumeration of the systems combinatorial methods are employed. This shows the possibility of relating the teaching of nomograms to the study of a type of Algebra more enlightening than that to which students are usually introduced. The viewpoint here involved may be illustrated by saying that alignment is the simplest case of a superposable plane, and that a dummy axis involves two superposable planes.

It may be of interest to analyse the references given at the beginning of "*Nomographie*". These are all in German, save for three of d'Ocagne's works which are excluded here, as well as six for which details are not given. The frequency distribution then according to size is

Number of pages	50	100	150	200	
Frequency of books	2	4	7	3	3.

This may give some idea of the part nomograms have played in German education and practice since 1918, when the earliest of these 19 volumes was published. Further references are given to German journals and to one Italian journal, and also to special German publications for nomograms used in chemistry, steel and machine construction, electrical work radiation technique, and aviation, artillery and defence problems.

The opening paragraph of the book, on functional scales from "stream and stress measurements", may be somewhat sophisticated for ordinary students and profoundly suspect with the scholar, but the theoretical aspect

of the subject is not at all neglected. Complaint might be made that the treatment of parallel scale nomograms gets into difficulty over the distances between the scales (so that each of the four practical nomograms of this type given has the middle scale exactly midway between the others!), and there may be a touch of opportunism about the suggestion on page 35 that as functional scales can be taken on the axes of Cartesian diagrams so they may be taken where convenient on any geometrical figure; but on the whole theoretical considerations are set out clearly enough. The author waxes almost lyrical over the qualities of alignment nomograms, though he does preserve his balance. One wonders that he does not feel the heavy overloading of the scales in his nomograms with numerals (on p. 67 elaborate numbering of a scale is repeated on the other side of the line); not trusting to the size of graduations for identification tends to obscure the character of the scale and the general numerical relations shown by the diagram as a whole. The treatment includes the general slide rule, and in the final chapter a functional scale is made to slide in any way over a diagram, this being a particular case of the use of sliding sheets, with an elaborate example of which involving seven variables the book concludes. The only omission noticed is the use of triangular rulings, which arise out of the theoretical treatment in Lambert's book.

Both these books would be of considerable help, each in its own way, in a real reconstruction of mathematical courses such as is needed.

JOHN MACLEAN.

G. H. HARDY AND E. M. WRIGHT: *An Introduction to the Theory of Numbers*. Oxford University Press (1938) 400 pages. 2s. sh.

One can count on one's fingers the number of books on the Theory of Numbers in the English language, and hence any new book on the subject is very welcome. When it comes from the hands of such an authority on the subject and inspiring writer as Professor G. H. Hardy, with the collaboration of Professor E. M. Wright, an expert worker in the field, every student of *Number Theory* will be simply delighted. Besides, this work has many features which secure for it a unique and prominent place among books on the subject.

The book is the outcome of lectures delivered by the authors in various Universities and offers an introduction to almost all the branches of Higher Arithmetic. By easy stages, the authors take the reader through various regions here and there giving glimpses of the many majestic peaks of the Number Theory, conquered or yet to be conquered. Whenever there is an opportunity, readers are taken on excursions to the allied fields of interest. The Chinese game of Nim, Bachet's problem of the weights, the construction of the 17-sided polygon, the Konig-Szuc's problem on the path of rays reflected on the sides of a square, etc. are discussed in detail. There is a digression even on a historical puzzle in one place. The book abounds in unsolved problems of intense interest. When the authors give here and there big numbers, saying "This is the greatest known prime", "this is the greatest number proved to be composite, but whose factors are unknown", "this is the value of $p(14031)$ " and so on, it produces a thrill in us.

Another interesting feature of the book is that several different proofs of varying interest are given for many important results, one result receiving as many as four different proofs. When an alternative proof requires an acquaintance with a different field, the authors take the trouble to develop the relevant portions of the alien field also.

Each chapter contains notes at the end, giving references to original sources, further developments in the topic, and sometimes a summary survey of the frontiers of the field. At the end a list of books for further reading and an index of special symbols and terms are also given.

The following rough summary of the contents of the book will give an idea of the variety and richness of the 'programme'.

Three Chapters (I, II, XXII) are devoted to the study of prime numbers. The main topics considered are the fundamental theorem of Arithmetic, Tchebychef's results on the order of the number of primes, Merten's theorem on $\sum 1/p$ and the numbers of Fermat and Mersenne.

In Chapter III, the properties of Farey series and Minkowski's theorem on the number of lattice points in a closed curve are established. Chapter XXIV contains proofs for three theorems of Minkowski concerning sets of n linear forms in n integral variables with real coefficients.

The authors devote chapters IV, X, XI to the study of irrational numbers and continued fractions. In Chapter X, elementary properties of simple continued fractions (periodic and nonperiodic) are investigated. Further, it deals with 'equivalent numbers', the series of Fibonacci and Lucas, and approximation by convergents. In Chapter XI, orders of approximation, the existence of transcendental numbers, the transcendence of e and π , and some theorems on continued fractions with bounded quotients are discussed.

Chapters V, VI, VII, VIII are concerned with congruences and residues. They contain proofs for theorems of Fermat, Wilson, Lagrange, Von Staudt, and Leudesdorf, the law of quadratic reciprocity, and Bauer's identical congruence; and their consequences.

Besides the usual discussion about the representation of numbers by decimals, Chapter IX contains 'tests for divisibility, certain interesting irrational numbers represented by decimals, some theorems on numbers with missing digits, and results on normal numbers.'

The study of quadratic fields occupies Chapters XII, XIV and XV. The fields $k(i)$ and $k(\rho)$ are studied in detail, and a few other fields are also considered in short. The fact that unique factorisation theorem is not universally true is clearly brought out. The finitude of 'Euclidean' and 'simple' quadratic fields is discussed and settled when the proof is easy. The proof for Lucas' test for the primality for the Mersenne number, M_{4^n+3} is also given.

In Chapter XIII the authors consider the Fermat's equation $x^n + y^n = z^n$, when $n=2, 3$ and 4. Richmond's theorem on the expression of a number as the sum of rational cubes is proved, and the equation ' $x^3 + y^3 = u^3 + v^3$ ' is completely solved.

In Chapters XVI, XVII, XVIII, the well-known arithmetic functions $\phi(n)$, $\mu(n)$, $d(n)$, $\sigma(n)$, $r(n)$ etc. are considered. Their generating functions, and their maximum, minimum and average orders are investigated. Many well-known identities including Ramanujan's are given. The multiplicative properties of the sums of Gauss, Ramanujan, and Kloostermann are established.

The theory of Partitions is considered in Chapter XIX. Several well-known identities in the theory due to Euler, Jacobi, and Roger and Ramanujan are proved. Certain congruence properties of the partition function due to Ramanujan are also established.

Waring's problem and the related topics form the subject-matter for Chapters XX and XXI. Chapter XX contains three of the four proofs given in the book for the theorem on the expression of a number as the sum of two squares, and three proofs for Lagrange's theorem on four squares. In Chapter XXI, upper bounds for $g(3)$ and $g(4)$, and lower bounds for $g(k)$ and $G(k)$ are evaluated. Further the function $v(k)$ (for the sums affected with signs) and Tarry's problem are also discussed.

Chapter XXIII deals with Kronecker's theorem on the simultaneous approximation of given irrationals to arbitrary numbers.

The book does not demand any great mathematical equipment on the part of the reader. From the Indian standpoint one can say that any intelligent mathematics graduate will be able to understand almost the whole of the book without outside help, while about threefourths of the book will be intelligible to a student of a degree class, having aptitude for mathematics.

The book is an intensely fascinating and very inspiring one. There is no doubt that it will inspire numerous youngmen to become ardent devotees to the 'queen' of the 'queen of all the sciences', and stimulate the interest of workers already in the field. Hence it is superfluous to say that every college library should possess at least one copy of this book of extraordinary charm.

The printing and get-up are attractive. I was able to note only one misprint: page 203, ' $2^{p-1} \equiv 0$ ' instead of ' $2^{p-1} \equiv 1$ '.

I wish to draw the attention of the authors to one or two minor points. Out of the four symbols borrowed from formal logic, apart from the symbol ' $\rightarrow$ ', the other three are used only in a dozen places or so. In view of the fact that keeping the number of new symbols at its minimum will be conducive to the easy reading of a book, I venture to suggest the omission of the above-mentioned three symbols in the future editions of the book.

The other suggestion is a general one. Is it not against the spirit of Mathematics to continue to call theorems by wrong names even when it has been established that they are wrong? If all the authors of books in Mathematics make up their minds to rectify the mistakes in current names of results, everything may be set alright in a few decades. Why should not authors rename in their books 'Farey series', 'Pell's equations', 'Cardan's solution' etc., at the cost of temporary inconvenience, and, act up to the spirit of the 'exact science'?

S. S. PILLAI.

MILNE THOMSON: *Theoretical Hydrodynamics*. Macmillan Co. London 31/6 net.

This book on Hydrodynamics by Prof. Milne-Thomson is a welcome addition to the small number of comprehensive treatises on the subject now available in English, especially in view of its several distinguishing features.

The treatment is based exclusively on vector methods and notation; the advantage of their use in deriving the general equations and in economising space besides giving a better physical insight into the phenomena discussed is specially seen in the chapters dealing with the Equation of Motion and Viscosity. And a very extensive use is made of the complex variable for the discussion of various problems of fluid motion.

One of the earlier chapters is devoted to the properties of vectors which are introduced without any reference to systems of co-ordinates, the subject-matter being arranged in such a way as to help those to whom vector manipulation may be unfamiliar. Again there is a chapter dealing the theory of functions of a complex variable up to contour integration and conformal representation. Thus in respect of the mathematical apparatus required, the book is fairly self-contained.

Two dimensional motion occupies some nine chapters in the book, separate chapters being devoted to general properties, streaming motion, aerofoils, sources and sinks, moving cylinders, Schwarz-Christoffel theorem and applications, jets and currents, and the wake behind a cylinder. One consequence of this arrangement is that allied problems come under different heads and could be discussed as illustrations of different methods. For instance, instead of treating all elliptic cylinder problems with the help of elliptic harmonics chosen to satisfy the necessary boundary conditions, streaming motion past an elliptic cylinder is obtained from the case of a circular cylinder by a Joukowski transformation, and the result translated into elliptic co-ordinates for the purpose of easy description of the flow; while the complex potential for a moving elliptic cylinder is obtained by a simple application of Morris's general method which is applicable whenever the co-ordinates of a point on the cross-section of a cylinder can be expressed in terms of a parameter. Here as elsewhere in the book is incorporated some account of recent work e.g. generalisations of Blasius's formula and Kutta-Joukowski theorem.

The chapter on the Wake includes an account of Levi-Civita's method of determining the flow past an obstacle, making this elegant method available for the first time in an English book. Two dimensional wave motion, which is the subject of chapter XIV, is all based on Cisotti's equation satisfied by the complex potential for waves of small height. There is a departure from the traditional presentation in the treatment of the motion of spheroids. Taking account of axial symmetry, the motion is studied with the help of the complex variable by means of the transformation $x+iw=c\sinh\xi$ and $x+iw=c\cosh\xi$ for the cases of the planetary and ovary spheroids. The quadratic transformation $x+iw=c\xi^2$ similarly gives the motion of a paraboloid of revolution.

In the traditional formulae giving the motion of the Ellipsoid, there is the unnecessary occurrence of the number 2 which is due to the notation

employed. This is easily avoided if the notation is slightly altered by taking α , α_0 etc. to represent half their usual values so that $\alpha_0 + \beta_0 + \gamma_0 = 1$; and the formulae gain considerably in appearance.

It is stated in the preface that the book has grown out of the author's lectures to classes in the Royal Naval College. The range covered is that usual in the introductory text-books, including chapters on Vertex motion and Viscosity. The exposition is clear and methodical, the needs of the student being borne in mind throughout. The book contains numerous illustrative diagrams and there are over 500 exercises collected into sets of examples at the end of each chapter. For easy reference, a detailed table of contents and a complete index are provided.

The book is thus assured of success as a text-book—though the price 31 s. 6d. is rather forbidding.

T. S. N.

Maurice CURIE et Maurice PROST: *Necessaire Mathematique*, Herman Fils, 1937; pp. 116; Actualites Scientifiques et industrielles No. 502 Price 20 fr.

The University of Paris offers a one year course for the award of a 'Certificat d'Etudes Physiques Chimiques et Biologiques' (briefly P. C. B.), this certificate being demanded of those who seek admission in the Faculty of Medicine. The revised programme of the P. C. B. presupposes a certain level of basic "Necessary Mathematics" which this book seeks to provide.

The book is intended also to provide the "necessary mathematics" for the S. P. C. N.—'Certificat d'Etudes Superieures Physiques Chimiques et Sciences Naturelles.' The level of mathematical equipment for this is somewhat higher and accordingly the additional matter intended for this course is printed in slightly smaller type. For example, the addition formulae trigonometric functions, the derivatives of inverse functions, and many of the proofs appear in small type.

In order to make the unpleasant mathematical pill palatable for the P. C. B. there is often an appeal to intuition in preference to a rigorous proof.

The several chapters deal with Algebra up to the solution of the quadratic, Trigonometry, graphs, calculus (Differential and integral including Maclaurin's Theorem and areas and volumes) and differential equations. There are 3 appendices dealing with small errors, probability and with dimensions in physical quantities.

A. NARASINGA RAO

BOOKS RECEIVED FOR REVIEW

Tsurusaburo TAKASU: *Differentialgeometrien in den Kugelräumen*, Bd. I
Konforme Differential-Kugelgeometrie von Liouville und Möbius
pp. 458, Maruzen Company, Ltd., Tokyo, Price 7.5 R. M.

Raziuddin SIDDIQI: *Lectures on Quantum Mechanics*, Vol. I, 1938 pp. 293,
Osmania University, Hyderabad (Deccan), Price Rs. 6

Raymond Clare ARCHIBALD: *Outline of the History of Mathematics*, 4th Ed.
Mathematical Association of America 1939, Price 50 cents.

Wilhelm BLASCHKE: *Ebene Kinematik*, Hamburger Mathematische
Einzelschriften 25 Heft, 1938, pp. 56, B. G. Teubner.

ANNOUNCEMENTS AND NEWS

Mr. Sulaiman Mahomed Kerawala, M.A., (Cantab.) Lecturer in Mathematics, Muslim University, Aligarh, has been elected a life member of the Society.

Mr. C. T. Rajagopal, M.A., Lecturer, Madras Christian College, has been elected a member of the Society.

The Hyderabad Conference in December 1939

An invitation to conduct the Eleventh Conference of the Indian Mathematical Society at Hyderabad (Deccan) on the 21st, 22nd and 23rd of December 1939 has been received from the Osmania University and accepted by the Society. The 10th session of the All-India Oriental Conference meets also in Hyderabad at the same time (20th to 23rd December) and the social functions and excursions of the two conferences will be arranged in common.

An influential reception committee has been constituted with the Pro-Vice-Chancellor of the Osmania University as Chairman. The local committee which will be in charge of the arrangements regarding the Conference session consists of Dr. M. R. Siddiqi, M.A., Ph.D. (Chairman), Mr. Khaja Mohi-nuddin, M.A. and Mr. Venkatachari, M.A.

The following programme has been provisionally fixed:

Thursday 21st December—

Forenoon: Opening ceremony, and the inaugural address.

Afternoon: Symposium on Generalised Geometry (including Relativity and Field theory); Popular lecture.

Friday 22nd December—

Forenoon : Reading of papers.

Afternoon : Symposium on Waring's problem ; Popular lecture.

Saturday 23rd December—

Forenoon : Reading of papers.

Afternoon : Business meeting ; Discussion on the Teaching of Mathematics ; Popular lecture.

Those wishing to contribute papers to the Conference are requested to send them along with 2 brief abstracts of each to A. Narasinga Rao, Annamalai nagar so as to be received by the middle of October.

The 27th session of the Indian Science Congress will be held at Madras from the 2nd to the 8th January 1940. Prof. A. C. Banerji, Head of the Mathematics Department, Allahabad University, Allahabad, has been nominated to be the President of the Mathematics Section. Papers should be sent to the General Secretary, Indian Science Congress Association, 92 Upper Circular Road, Calcutta, so as to reach him by the 15th September 1939 along with 3 typed abstracts of each.

The third session of the Indian Statistical Conference will meet in Mysore on the 4th and 5th January 1940.

Mr. S. Subramaniam, M.A., Lecturer in Mathematics, Annamalai University, has been appointed Statistician to the Economic adviser to the Government of India in place of Dr. P. V. Sukhatme, M.A. Ph.D. (London) who has accepted the post of Assistant Professor of Vital Statistics in the All-India Institute of Public Health and Hygiene, Calcutta.

QUESTIONS FOR SOLUTION

1770. (D. R. KAPREKAR)—The fractions $1/487$ and also $1/(487)^2$ when expressed as decimals have the same number of recurring figures namely 486. Required to find all numbers N such that $1/N$ and $1/N^2$ have the same number of recurring figures in their decimal representation.

1771. (KAIKOBAD F. VAKIL)—Taking the given number 37, 781,456, 166, 601, 770 as step number zero, find after what number of steps we shall get a perfect square by adding to it 50, 952, 413, 280, 206, 183 ; 50, 952, 413, 380, 206, 185 ; 50, 952, 413, 380, 206, 187 ; etc. successively which are in A.P. with a common difference 2.

e.g.	37781456166601770 50952413380206183	Step Number 0.
	88733869546807953 50952413380206185	... (1)
	139686282927014138 50952413380206187	... (2)
	190638696307220325 etc.	... (3)

1772. (B. R. VENKATARAMAN)— C_i ($i=1, 2, \dots, 4$) are 4 circles in a plane and S_i ($i=1, \dots, 4$) are the four circles which are respectively orthogonal to sets of three circles chosen from C_i . If the inverses of a point P w. r. t. the circles C_i are concyclic, prove that the inverses of P w. r. t. the circles S_i are also concyclic.

1773. (G. V. RAMACHANDRAN and B. RAMAMURTI)— l_1, l_2, l_3, l_4 are four straight lines in a plane. L_{rs} is the point of intersection of the lines l_r, l_s . If P be any point in the plane show that

(1) there is a conic Σ touching the four lines, for which PL_{21}, PL_{24} and PL_{31}, PL_{34} form two pairs of conjugate lines

(2) the points P and L_{14} are conjugate with respect to Σ

(3) If PQ, PR be any pair of conjugate lines of Σ

$$P(QL_{13}RL_{34}) = P(QL_{13}RL_{24})$$

(4) Given the pairs of points A_1A_2, A_2A_4 and another point P all lying in a plane, the pairs of lines PQ, PR such that

$$P(QA_1RA_2) = P(QA_3RA_4)$$

are conjugate with respect to a fixed conic Σ touching the lines $A_1A_3, A_1A_4, A_2A_3, A_2A_4$.

1774. (S. S. PILLAI)— n beads are connected in the form of a chain by means of $n-1$ links of which r are detachable. By removing at most t of these detachable links at a time, we should be able to form groups containing any number of beads from 1 to n . Let $f(t, n)$ denote the least value of r so that the above may be possible. It is easy to see that

$$f(1, n) = [(n+1)/2] \text{ and } f(t, n) = t, \text{ when } t = [(\log n)/(\log 2)] + 1.$$

What is the value of $f(t, n)$ for intermediate values of t ?

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ERRATA FOR VOLUME VI

<i>Page</i>	<i>line</i>	<i>for</i>	<i>Read</i>
68	9B†	13345678987654321	12345678987654321
74	6B	$k = 0$	$\kappa = 0$
77	2	$\sum \frac{z^r}{r!} = 0$	$\sum \frac{z^r}{r!}$
83	12	$\frac{1}{2} AB \cdot DE$	$\frac{1}{2} AB \cdot AE$
99	17-32	Remove the first 5 sentences of § 3 (ending with the words "Circular points at infinity") and replace them by the following: "Any two concurrent tetrads of the Miquel-Clifford configuration have 2 circles in common. Let P and Q be the limiting points of the coaxal system they determine. The two cyclics associated with these tetrads have in common as we have seen, the 18 points of the configuration and also P, Q. Hence they must be identical."	

† Read "Ninth line from the bottom of the page."
